

Artificial Boundary Condition Processing Technique for Phased-Resolved Wave Surface Reconstruction

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Abstract

At present, the measurement of the near wave field of ships mostly relies on shipborne radar. The commonly used shipborne radar is incoherent and cannot obtain information on wave surface velocity. Therefore, the mathematical model of wave reconstruction is remarkably complex. As a new type of radar, coherent radar can obtain the radial velocity of the wave surface. Most wave surface reconstruction methods that use wave velocity are currently based on velocity potential. The difficulty of these methods lies in determining the initial value of the velocity integral. This paper proposes a wave surface reconstruction method based on an artificial boundary matrix. Numerical simulation data of regular and short-crest waves are used to verify the accuracy of this method. Simultaneously, the reconstruction stability under different wave velocity measurement errors is analyzed. The calculation results show that the proposed method can effectively realize the reconstruction of wave field.

Keywords Coherent radar; Wave velocity field; Artificial boundary matrix; Wave surface reconstruction; Calculation stability

1 Introduction

When ships sail on the sea, the six-degree-of-freedom (6-DOF) motion of the ships has considerable effects on their safety and operation. The wave is one of the main environmental factors that cause the 6-DOF motion of ships. Therefore, obtaining the near wave field of ships is crucial to ensure their safety.

Obtaining wave field measurements of the ship's loca-

tion by deploying buoys or Acoustic doppler current profiler (ADCP) when the ship is sailing in the sea is not feasible. Additionally, shipborne radars experience shadowing effects in the vicinity of the ship, preventing direct measurement of wave field data in the ship's near field. Two methods used to obtain the near wave field of ships are currently available. The first method involves reconstructing the wave according to the time series of ship motion using the Kalman filter (Nielsen, 2019) and neural network (Dug, 2019). However, the limitation of this method lies in obtaining only the wave at the ship position and failing to measure the wave at a distance from the ship.

The second method involves reconstructing the wave surface according to the wave surface radar image of shipborne radar. This reconstruction uses radar to sweep waves in a large area to obtain wave information. At present, incoherent radar is widely used. Radar image sequencing is the most commonly used method in the wave surface reconstruction method based on incoherent radar. First, the three-dimensional fast Fourier transform (3D-FFT) is utilized to convert the image sequence into an image spectrum, and the dispersion relationship is used to filter the image spectrum to extract related wave information. The modulation transfer function (MTF) determined by experience is then utilized to convert the filtering spectrum into a wave spectrum. Nieto-Borge et al. (2004) proposed the method of using 3D-FFT to obtain the wave information from the radar image sequence information. Dankert et al. (2004) eliminated the influence of the angle on the imaging by comparing the

Article Highlights

- In order to ensure the operation safety of ship structures at sea and improve the operation efficiency, it is necessary to accurately monitor the wave environment.
- The phased-resolved wave surface reconstruction method of coherent radar based on wave is studied.
- The velocity difference method is used to solve the problem of constant term error in integral solution.
- The calculation results of regular wave and irregular wave under different sea states are compared.

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angle of each pixel in the image with respect to the shooting angle of view. However, their wave surface reconstruction based on the MTF method has low credibility due to the influence of multiple factors such as radar and sea surface noise. Meanwhile, when the radar sweep angle is low, the occlusion of the wave crest to the trough area will further complicate the process of pushing the MTF based on the physical method. Barstow et al. (2005) compared the wave height retrieved from the radar image with that measured by the wave finder. They found that the wave phases between the two are relatively consistent, but a certain difference exists between the wave heights. Aragh et al. (2008) proposed a method to realize wave height inversion through the conversion function between the wave model to be predicted and the radar image information. Based on the least-square method, Wang et al. (2009) used the fitting method of radar image information and experimental data measured by buoys to invert the wave field. Li (2013) used the Laplace operator to preprocess the noise in ocean radar images, eliminated the abnormal point data, and used the least squares method for fitting, which effectively improved the inversion accuracy of wave surface elevation.

Compared with incoherent radar, the best feature of coherent radar lies in its capability to obtain wave surface velocity field. Nwogu and Lyzenga (2010) integrated the radial velocity of the coherent radar to reconstruct the numerical solution of the wave surface velocity potential and then calculated the wave surface accordingly. Serafino et al. (2009) used the radial Doppler velocity obtained by the coherent radar to directly reconstruct the wave surface and verified the feasibility of this method under deep water conditions. Carrasco et al. (2017) proposed a semiempirical method for estimating the effective wave height. After a large amount of data analysis, the results revealed that the effective wave height has a linear relationship with the root mean square of the Doppler velocity. This method directly acquires the effective wave height from the velocity information. The method also avoids the use of MTF and obtains an accurate wave height estimate.

When integrating the velocity to obtain the velocity potential, determining the initial value of the integral is difficult. In practical engineering applications, directly obtaining $\phi(r_{\min})$ through actual measurements is difficult. This difficulty has a crucial impact on the accuracy of the wave surface reconstruction. This paper proposes a wave surface reconstruction method based on an artificial boundary matrix to achieve wave surface reconstruction, which avoids the determination of integral initial values. The numerical simulation data are used to verify this method. Section 2 explains the basic theory of the reconstruction method. Section 3 describes the wave conditions of the numerical simulation. Section 4 analyzes the reconstruction results of regular and short-crest waves. Section 5 examines the influence of measurement error on velocity.

2 Theory of reconstruction method

The difference between coherent and incoherent radars lies in the variation of the magnetron system in the equipment. The phase of the electromagnetic wave signal generated by the incoherent radar is irrelevant, and the Doppler frequency shift information of the moving object cannot be obtained. Meanwhile, accurate Doppler frequency shift information can be obtained when the phase of the electromagnetic wave signal generated by the coherent radar is relevant. Coherent radar can obtain the velocity information of the wave surface through the Doppler frequency shift information of the wave echo signal.

In the potential flow theory, the first-order linear definite solution condition of fluid velocity potential ϕ can be expressed as follows:

$$\begin{cases} \Delta\phi = 0, & \text{in the fluid domain} \\ \frac{\partial\phi}{\partial t} = -g\eta, & \text{on } z = 0 \\ \frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial z}, & \text{on } z = 0 \\ \frac{\partial\phi}{\partial z} = 0, & \text{on } z = -h \end{cases} \quad (1)$$

where h is the water depth, g is the gravity acceleration, and η is the fluid-free surface. Assuming the periodic waves in time and space, the following analytical Airy solution is obtained for the free surface elevation and velocity potential.

$$\eta = A \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \quad (2)$$

$$\phi = \frac{Ag}{\omega} \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) \frac{\cosh[k(z+h)]}{\cosh(kh)} \quad (3)$$

where A is the wave amplitude, $\mathbf{k} = (k \cos \theta, k \sin \theta)$ is the wave number vector, θ is the direction angle, ω is the wave frequency and $\mathbf{r} = (x, y)$ is the spatial location.

In this paper, the water depth is assumed to be infinite, and the wave satisfies the dispersion relation of infinite water depth.

$$\omega^2 = gk \quad (4)$$

The basic assumption for the wave field generally indicates that irregular waves are superimposed by multiple regular waves of different frequencies, amplitudes, and phases ϕ . Based on the linear superposition of waves, Mohapatra et al. (2019) added the influence of the second-order term to realize the simulation of second-order internal waves. This paper uses only the linear wave theory.

$$\phi = \sum_{i=1}^N \frac{a_i g}{\omega_i} \sin[k_i(x \cos \theta_i + y \sin \theta_i) - \omega_i t + \phi_i] \quad (5)$$

$$\eta = \sum_{i=1}^N a_i \cos[k_i(x \cos \theta_i + y \sin \theta_i) - \omega_i t + \phi_i] \quad (6)$$

or

$$\phi = \sum_i \sum_j \frac{a_i g}{\omega_i} \cos [k_i (x \cos \theta_j + y \sin \theta_j) - \omega_i t + \varphi_i] \quad (7)$$

$$\eta = \sum_i \sum_j a_i \cos [k_i (x \cos \theta_j + y \sin \theta_j) - \omega_i t + \varphi_i] \quad (8)$$

When the spatial position is expressed in polar coordinates, the linear velocity potential and wave surface equation at the free surface ($z=0$) can be expressed as follows:

$$\phi = \sum_i \frac{a_i g}{\omega_i} \sin [k_i (r \cos \beta \cos \theta_i + r \sin \beta \sin \theta_i) - \omega_i t + \varphi_i] \quad (9)$$

$$\eta = \sum_i a_i \cos [k_i (r \cos \beta \cos \theta_i + r \sin \beta \sin \theta_i) - \omega_i t + \varphi_i] \quad (10)$$

or

$$\phi = \sum_i \sum_j \frac{a_i g}{\omega_i} \cos [k_i (r \cos \beta \cos \theta_j + r \sin \beta \sin \theta_j) - \omega_i t + \varphi_i] \quad (11)$$

$$\eta = \sum_i \sum_j a_i \cos [k_i (r \cos \beta \cos \theta_j + r \sin \beta \sin \theta_j) - \omega_i t + \varphi_i] \quad (12)$$

where a_i , ω_i , k_i , θ_i , and φ_i represent the wave amplitude, frequency, wave number, direction angle, and phase angle of each cosine wave, respectively, r is the radius in polar coordinates, and β is the angle in polar coordinates.

The two short-crest wave generation methods are single and double superposition methods. The main difference lies in whether the direction angle is connected to the frequency.

Simultaneously, the corresponding velocity field of the wave surface can be obtained. Considering the radial velocity field that can only be obtained by coherent radar, only the expression of the radial velocity field is written. Herein, only the single superposition method is taken as an example:

$$u_r = \frac{\partial \phi}{\partial r} = \sum_i \frac{a_i g}{\omega_i} (\cos \beta \cos \theta_i + \sin \beta \sin \theta_i) \times \cos [k_i (r \cos \beta \cos \theta_i + r \sin \beta \sin \theta_i) - \omega_i t + \varphi_i] \quad (13)$$

The radial velocity and the inverted velocity potential

are obtained by the coherent radar and integration, respectively, and the inverted wavefront is then obtained (Nwogu and Lyzenga, 2010).

$$\phi(r) = \int_{r_{\min}}^r u_r r dr + \phi(r_{\min}) \quad (14)$$

where r is the radial distance. The above-mentioned integral formula for inverting the velocity potential (Eq. (14)) is theoretically feasible. However, in actual operation, this integral formula has several problems, such as the determination of the initial value $\phi(r_{\min})$ of the integral. Therefore, another method can be used to calculate the velocity potential.

The velocity is the derivative of the velocity potential, and the derivative can be approximated by the difference method. Thus, this research uses the idea of deriving the velocity potential difference to perform the inversion. The five-point difference is used as a method of numerical derivation as follows:

$$f(x_0) = \frac{1}{12\Delta x} [-25f(x_0) + 48f(x_1) - 36f(x_2) + 16f(x_3) - 3f(x_4)] \quad (15)$$

$$f(x_1) = \frac{1}{12\Delta x} [-3f(x_0) - 10f(x_1) + 18f(x_2) - 6f(x_3) + f(x_4)] \quad (16)$$

$$f(x_2) = \frac{1}{12\Delta x} [f(x_0) - 8f(x_1) + 8f(x_3) - f(x_4)] \quad (17)$$

$$f(x_3) = \frac{1}{12\Delta x} [-f(x_0) + 6f(x_1) - 18f(x_2) + 10f(x_3) + 3f(x_4)] \quad (18)$$

$$f(x_4) = \frac{1}{12\Delta x} [3f(x_0) - 16f(x_1) + 36f(x_2) - 48f(x_3) + 25f(x_4)] \quad (19)$$

The edge difference scheme is used for the two grid points near the boundary, and the central difference scheme is used for the other grid points. Therefore, the entire reconstruction process can be written as Eq. (21).

$$\begin{bmatrix} -25 & 48 & -36 & 16 & -3 \\ -3 & -10 & 18 & -6 & 1 \\ & 1 & -8 & 0 & 8 & -1 \\ & & 1 & -8 & 0 & 8 & -1 \\ & & & \dots & \dots & \dots & \dots \\ & & & & 1 & -8 & 0 & 8 & -1 \\ & & & & & 1 & -8 & 0 & 8 & -1 \\ & & & & & & -1 & 6 & -18 & 10 & 3 \\ & & & & & & & 3 & -16 & 36 & -48 & 25 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \dots \\ \phi_{N-3} \\ \phi_{N-2} \\ \phi_{N-1} \\ \phi_N \end{bmatrix} = \begin{bmatrix} u_{r_0} r_0 \\ u_{r_1} r_1 \\ u_{r_2} r_2 \\ u_{r_3} r_3 \\ \dots \\ u_{r_{N-3}} r_{N-3} \\ u_{r_{N-2}} r_{N-2} \\ u_{r_{N-1}} r_{N-1} \\ u_{r_N} r_N \end{bmatrix} \quad (20)$$

$$\begin{bmatrix}
 -25 & 48 & -36 & 16 & -3 & & & \\
 -3 & -10 & 18 & -6 & 1 & & & \\
 & 1 & -8 & 0 & 8 & -1 & & \\
 & & 1 & -8 & 0 & 8 & -1 & \\
 & & & \dots & \dots & \dots & \dots & \\
 & & & \dots & \dots & \dots & \dots & \\
 & & & & 1 & -8 & 0 & 8 & -1 \\
 & & & & & 1 & -8 & 0 & 8 & -1 \\
 & & & & & & -1 & 6 & -18 & 10 & 3 \\
 & & & & & & & 3 & -16 & 36 & -48 & 25
 \end{bmatrix}^{-1}
 \begin{bmatrix}
 u_{r_0} r_0 \\
 u_{r_1} r_1 \\
 u_{r_2} r_2 \\
 u_{r_3} r_3 \\
 \dots \\
 \dots \\
 u_{r_{N-3}} r_{N-3} \\
 u_{r_{N-2}} r_{N-2} \\
 u_{r_{N-1}} r_{N-1} \\
 u_{r_N} r_N
 \end{bmatrix}
 =
 \begin{bmatrix}
 \phi_0 \\
 \phi_1 \\
 \phi_2 \\
 \phi_3 \\
 \dots \\
 \dots \\
 \phi_{N-3} \\
 \phi_{N-2} \\
 \phi_{N-1} \\
 \phi_N
 \end{bmatrix} \quad (21)$$

where u_r represents the radial velocity field measured by a coherent radar, ϕ represents the reconstruction velocity potential obtained, and N is the number of radial velocities in the numerical calculation process, which is taken as 300 in this paper. The index $i=0, 1, \dots, N$ corresponds to the radar selection range of 100–400 m, where $i=0$ is the closest, and $i=N$ is the furthest. The wave surface can be calculated using the free surface condition after obtaining the velocity potential.

Notably, the coefficient matrix in the formula is obtained by the five-point difference formula. This matrix is normal when calculating the speed in the forward direction through the velocity potential. However, calculating the velocity potential through the velocity inverse can lead to problems. The main problem lies in the smaller rank of the difference

coefficient matrix than the number of rows of the matrix, thus making it impossible to find the inverse matrix of the coefficient matrix. This coefficient matrix cannot change the rank by varying the combination of differences in the rows of the matrix (for example, the first, second, and third rows use five-, four-, and three-point differences, respectively). During the differential solution, one row of missing data is observed in the differential calculation compared to the original result due to the limited boundary. Considering that the boundary is limited when the velocity potential of the innermost layer is solved, the first row (the innermost layer) is taken as the artificial boundary, and its coefficient is changed to 1. That is, the velocity potential in this layer is equal to the velocity; therefore, the matrix cannot be inverted.

$$\begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & & & \\
 -3 & -10 & 18 & -6 & 1 & & & \\
 & 1 & -8 & 0 & 8 & -1 & & \\
 & & 1 & -8 & 0 & 8 & -1 & \\
 & & & \dots & \dots & \dots & \dots & \\
 & & & \dots & \dots & \dots & \dots & \\
 & & & & 1 & -8 & 0 & 8 & -1 \\
 & & & & & 1 & -8 & 0 & 8 & -1 \\
 & & & & & & -1 & 6 & -18 & 10 & 3 \\
 & & & & & & & 3 & -16 & 36 & -48 & 25
 \end{bmatrix}^{-1}
 \begin{bmatrix}
 u_{r_0} r_0 \\
 u_{r_1} r_1 \\
 u_{r_2} r_2 \\
 u_{r_3} r_3 \\
 \dots \\
 \dots \\
 u_{r_{N-3}} r_{N-3} \\
 u_{r_{N-2}} r_{N-2} \\
 u_{r_{N-1}} r_{N-1} \\
 u_{r_N} r_N
 \end{bmatrix}
 =
 \begin{bmatrix}
 \phi_0 \\
 \phi_1 \\
 \phi_2 \\
 \phi_3 \\
 \dots \\
 \dots \\
 \phi_{N-3} \\
 \phi_{N-2} \\
 \phi_{N-1} \\
 \phi_N
 \end{bmatrix} \quad (22)$$

3 Inversion condition

The method is verified by numerical simulations due to the lack of measured data. Considering computational efficiency, a linear model is selected as the wave model for numerical simulation. The velocity direction measured by the coherent radar is radial; thus, only the radial velocity field information is necessary in the simulation.

In this paper, regular and short-crest waves are considered, and the working conditions used in the numerical simulation verification of coherent radar wave measurement are shown in Table 1.

Table 1 Numerical simulation conditions

Wave	Ocean wave spectra	Sea state	Significant wave (m)	Characteristic period (s)
Regular wave	—	—	3.5	8.5
			5.5	9.5
			7.5	10.5
Short-crest waves	ITTC spectrum	5	3.5	8.5
		6	5.5	9.5
		7	7.5	10.5

4 Analysis of reconstruction results

4.1 Regular wave reconstruction results

Under the framework of linear theory, irregular waves are usually considered to be formed by the linear superposition of a series of regular waves of different frequencies and amplitudes. Numerical simulation under regular wave conditions can intuitively reflect and verify the performance of the method.

First, the regular wave with a wave height of 3.5 m is reconstructed. Figures 1(a) and 1(c) below show the theoretical wave surface and reconstructed velocity potentials, respectively. The figures reveal that the reconstructed results of the velocity potential are highly consistent with the theoretical results. The theoretical wave surface and the reconstructed wave corresponding to the velocity potential are shown in Figures 1(c) and 1(d), respectively.

The inversion error of 3.5 m wave height is shown in Figure 1(e) to intuitively compare the error of the wave inversion method of coherent radar images. The figures reveal that in the case of a 3.5 m regular wave, the reconstructed model has high accuracy, and the absolute error of the wave surface is controlled within 10^{-4} – 10^{-5} m, which verifies the accuracy of the reconstructed model.

Figures 2(a) and 2(c) show the wave surface and inversion velocity potentials of regular waves with a wave height of 5.5 m, respectively. The figures reveal that the inversion result image of velocity potential is almost consistent with the theoretical result image. The theoretical wave surface corresponding to the velocity potential and the inversion wave is shown in Figures 2(b) and 2(d), respectively, and the accuracy of the reconstructed wave surface is also excellent.

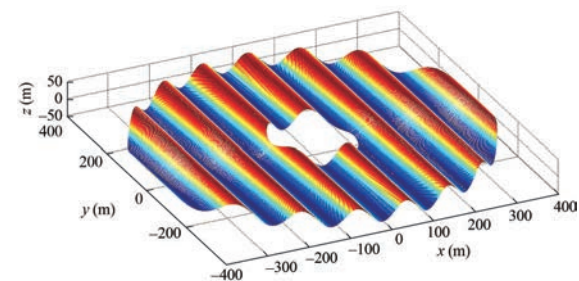
The absolute error in Figure 2(e) shows that the overall error of the inversion effect at 5.5 m is only slightly different from that at 3.5 m, which is controlled within 10^{-4} – 10^{-5} m, which verifies the accuracy of the inversion model.

Figures 3(a) and 3(c) depict the theoretical and inversion velocity potentials, respectively, for a regular wave with a wave height of 7.5 m. The inversion results of the velocity potential closely match the theoretical results. The corresponding theoretical wave surface to the velocity potential and the inversion wave are presented in Figures 3(b) and 3(d), respectively.

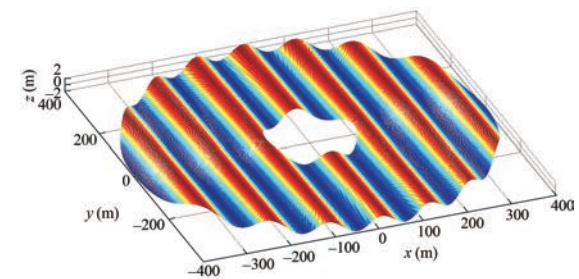
The absolute error of the wave surface is examined, as shown in Figure 3(e), to further evaluate the inversion performance. The reconstruction error is assessed for a wave height of 7.5 m, and the error magnitude is on the order of 10^{-4} . These findings demonstrate that the reconstruction model exhibits high accuracy, thereby confirming the validity of the reconstruction approach.

The above numerical results (Figures 1, 2 and 3) under the condition of regular waves with different amplitudes are consistent, which shows that the numerical algorithm developed

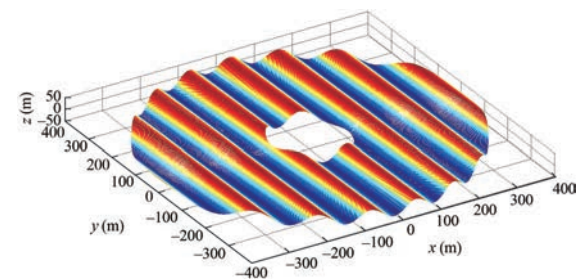
in this paper is correct and feasible. This study provides a research basis for the subsequent validation of coherent radar algorithms under short-crest wave conditions.



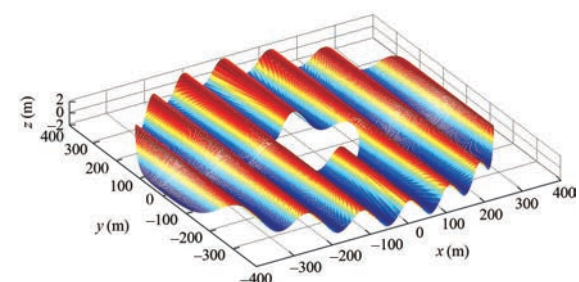
(a) Original velocity potential



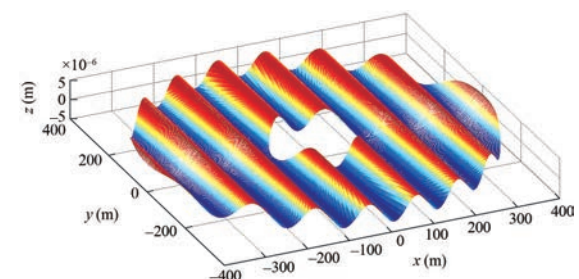
(b) Original wave surface



(c) Reconstructed velocity potential

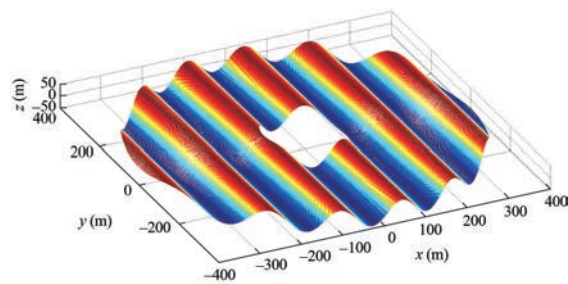


(d) Reconstructed wave surface

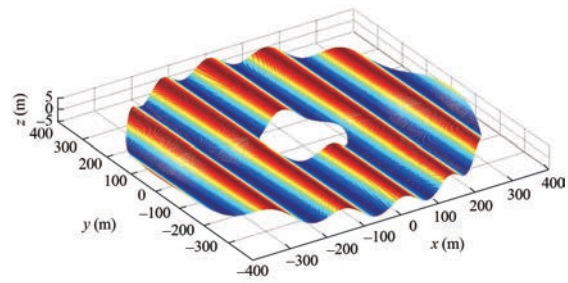


(e) Reconstruction error

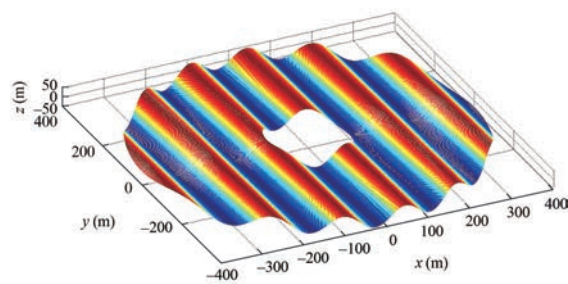
Figure 1 Reconstruction result of regular wave ($H = 3.5$)



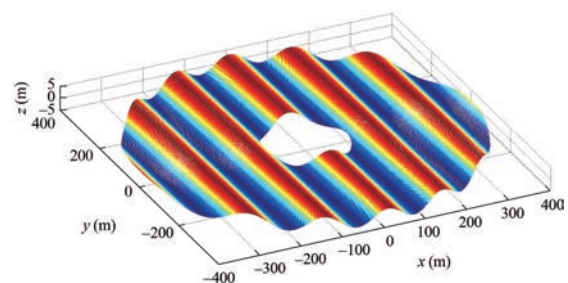
(a) Original velocity potential



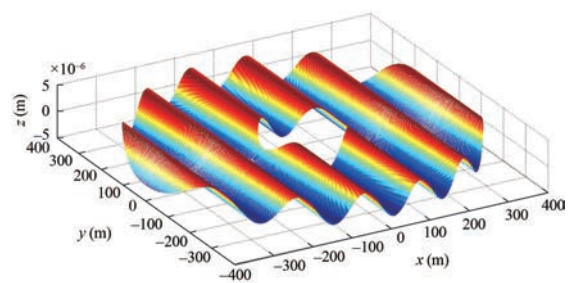
(b) Original wave surface



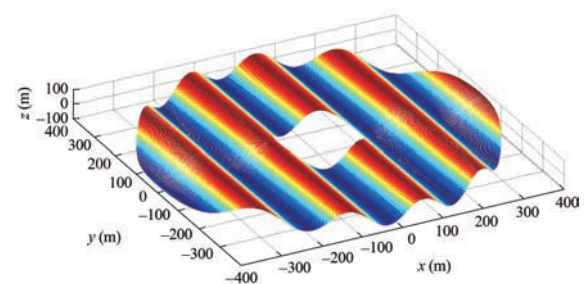
(c) Reconstructed velocity potential



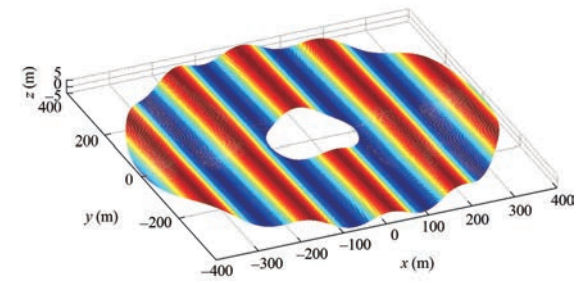
(d) Reconstructed wave surface



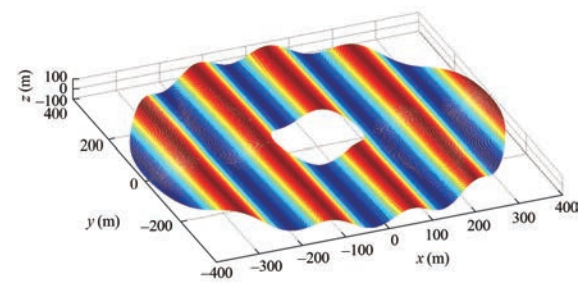
(e) Reconstruction error

Figure 2 Reconstruction result of regular wave ($H = 5.5$)

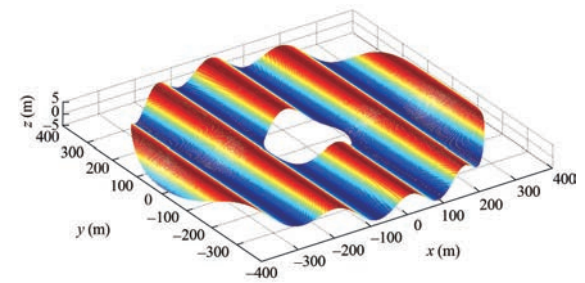
(a) Original velocity potential



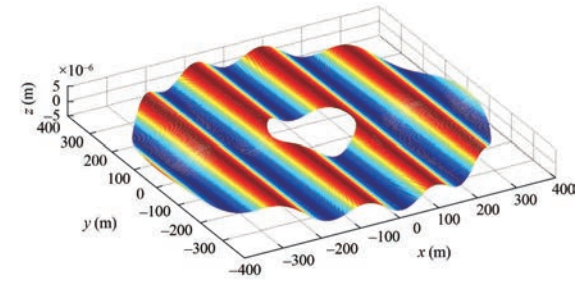
(b) Original wave surface



(c) Reconstructed velocity potential



(d) Reconstructed wave surface



(e) Reconstruction error

Figure 3 Reconstruction result of regular wave ($H = 7.5$)

4.2 Short-crest reconstruction results

The short-crest wave is the wave in the actual ocean; therefore, conducting the inversion simulation of the short-crest wave is necessary. Considering the computational efficiency, the linear wave model is used to generate the short-crest wave surface. The velocity potential, wave surface, and r -direction velocity field are respectively shown as follows:

$$\phi = \sum_i \sum_j \frac{a_i g}{\omega_i} \cos [k_i (r \cos \beta \cos \theta_j + r \sin \beta \sin \theta_j) - \omega_i t + \varphi_i] \quad (23)$$

$$\eta = \sum_i \sum_j a_i \cos [k_i (r \cos \beta \cos \theta_j + r \sin \beta \sin \theta_j) - \omega_i t + \varphi_i] \quad (24)$$

$$u_r = \frac{\partial \phi}{\partial r} = \sum_i \frac{a_i g}{\omega_i} (\cos \beta \cos \theta_i + \sin \beta \sin \theta_i) \times \cos [k_i (r \cos \beta \cos \theta_i + r \sin \beta \sin \theta_i) - \omega_i t + \varphi_i] \quad (25)$$

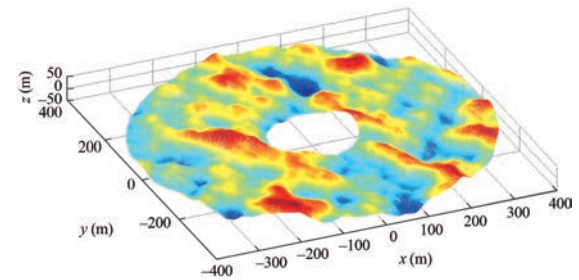
The velocity potentials of the theoretical wave surface and the inversion results under sea state 5 are presented in Figures 4(a) and 4(c), respectively. Notably, the reconstructed velocity potentials exhibit remarkable agreement with the theoretical counterparts. The corresponding theoretical wave surfaces corresponding to the velocity potential and the reconstructed waves are illustrated in Figures 4(b) and 4(d), respectively, displaying strong consistency with the velocity potential outcomes. The absolute error of the wave surface is examined in Figure 4(e) to further assess the effectiveness of the reconstruction approach. Assessment results revealed that the reconstruction error under sea state 5 is consistently on the order of 10^{-4} , emphasizing the high accuracy of the reconstruction model. These results further validate the accuracy and reliability of the proposed reconstruction model.

Figures 5(a) and 5(c) show the theoretical wave surface and reconstructed velocity potentials under sea state 6, respectively. The figures show almost no difference in the velocity potential between the two. Figures 5(b) and 5(d) are the theoretical wave surface and the reconstructed wave, respectively. The comparison results agree with the theoretical results.

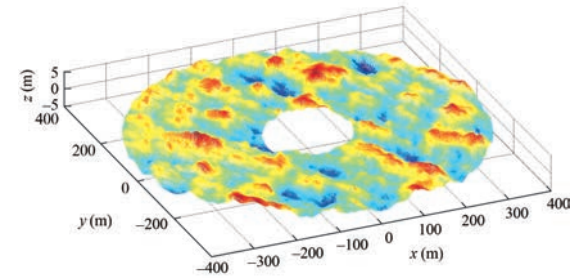
The absolute error of the inversion method under sea state 6 is illustrated in Figure 5(e) to facilitate a comprehensive comparison of the wave inversion error using coherent radar imagery. Notably, the figure demonstrates the high accuracy of the inversion algorithm, especially under sea state 6.

The theoretical wave surface velocity potential (Figure 6(a)) and the reconstructed velocity potential (Figure 6(c)) under sea state 7 exhibit minimal discrepancies, exhibiting a high degree of similarity between the two velocity potentials. Figures 6(b) and 6(d) show the theoretical and reconstructed wave surfaces, respectively. Notably, the comparison reveals a significant degree of concordance between the images

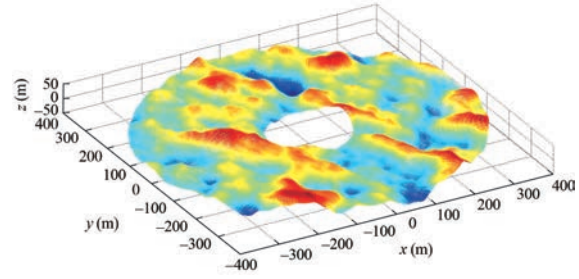
of the theoretical and reconstructed wave surfaces, which closely aligns with the results obtained from the velocity potential analysis.



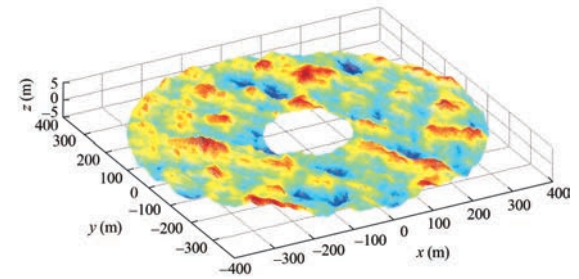
(a) Original velocity potential



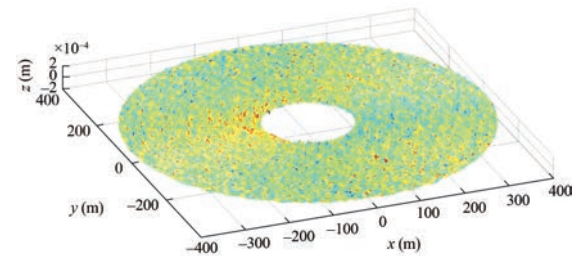
(b) Original wave surface



(c) Reconstructed velocity potential

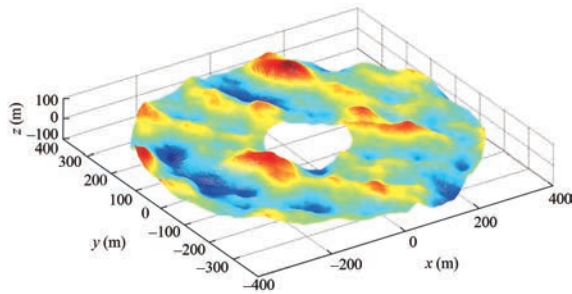


(d) Reconstructed wave surface

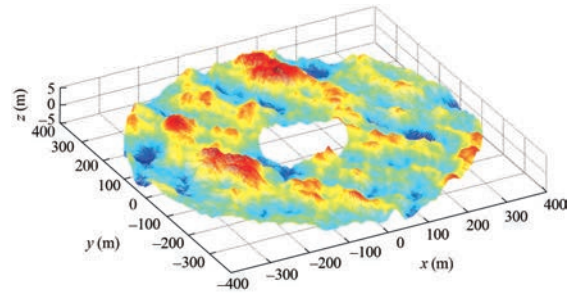


(e) Reconstruction error

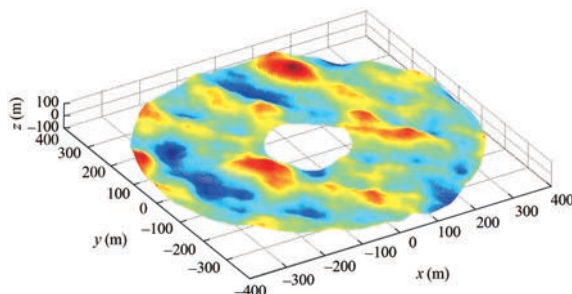
Figure 4 Reconstruction result of short-crest wave (sea state 5)



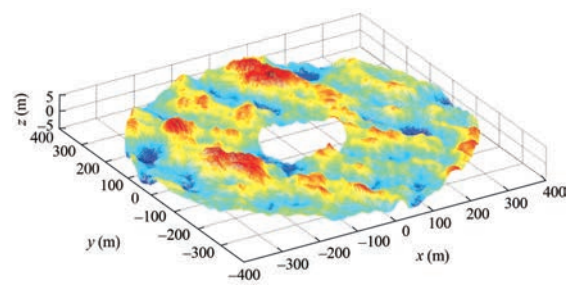
(a) Original velocity potential



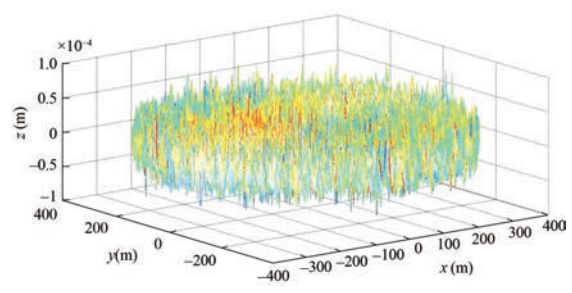
(b) Original wave surface



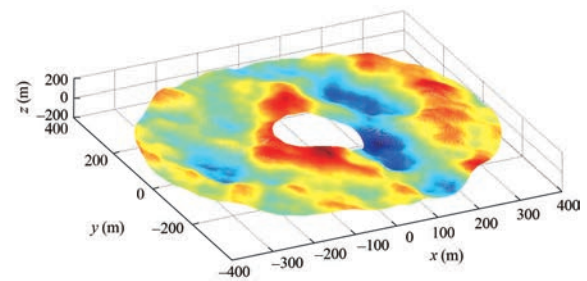
(c) Reconstructed velocity potential



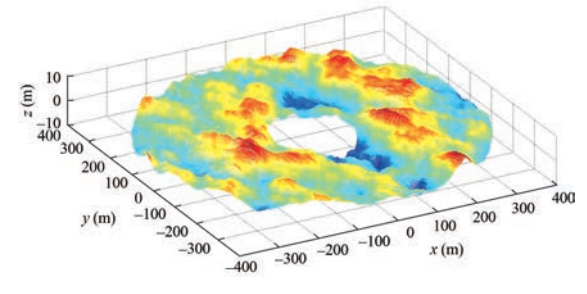
(d) Reconstructed wave surface



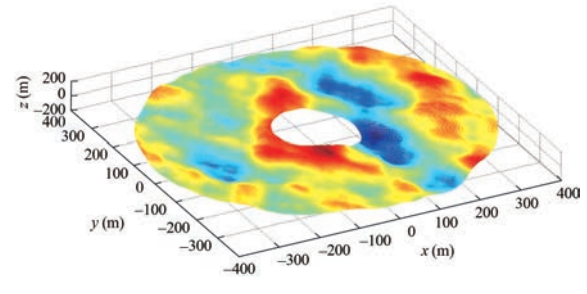
(e) Reconstruction error

Figure 5 Reconstruction result of short-crest wave (sea state 6)

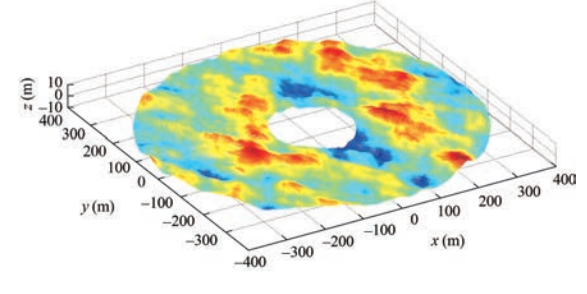
(a) Original velocity potential



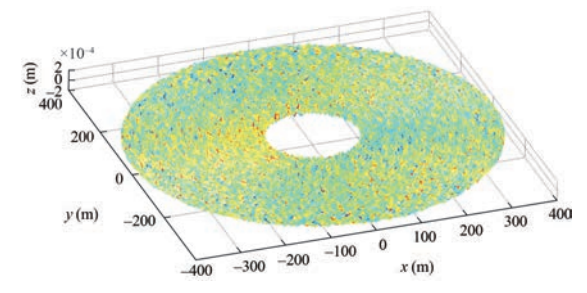
(b) Original wave surface



(c) Reconstructed velocity potential



(d) Reconstructed wave surface



(e) Reconstruction error

Figure 6 Reconstruction result of short-crest wave (sea state 7)

Figure 6(e) intuitively shows the effect of the wave inversion method through the absolute error of the wave surface. The figure reveals that the overall wave surface error remains within the range of 10^{-4} – 10^{-5} m under sea state 7, and the error is substantially small. These findings prove that the wave surface elevation inversion algorithm proposed in this paper has high inversion accuracy.

The root mean square statistics of inversion error under sea states 5–7 are given in Table 2 to further analyze the accuracy of the wave reconstruction algorithm under different sea states. Regardless of the influence of measurement errors, the table shows that the algorithm proposed in this paper demonstrates a high level of inversion accuracy, with error magnitudes controlled within the range of 10^{-5} m. The aforementioned results validate the feasibility and accuracy of the algorithm for inverting the elevation of deterministic irregular wave surfaces.

Table 2 Reconstructed error of short-crest wave

Sea state	Root mean square error (m)
5	1.42×10^{-5}
6	1.78×10^{-5}
7	2.00×10^{-5}

5 Stability analysis

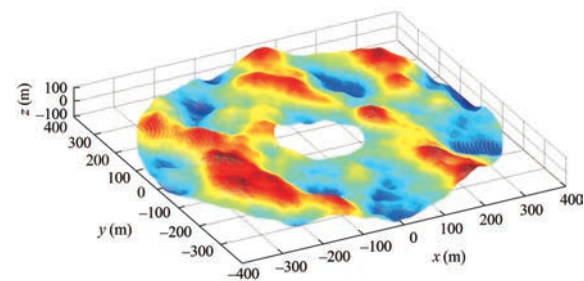
Some measurement errors are encountered when coherent radar is used to measure the wave surface velocity field in practice, which requires the accuracy and stability of the reconstruction model. Therefore, discussing the influence of velocity error on the performance of reconstructed results is necessary.

This paper selects sea state 7 as the calculation sea state to observe the influence of velocity error on the reconstructed accuracy and takes a series of velocity measurement errors (0.1%, 0.5%, 1.0%, and 2.0%) to analyze the reconstructed results under different errors and study the accuracy stability of the reconstruction model. In the process of numerical simulation, the velocity error is considered by the following methods

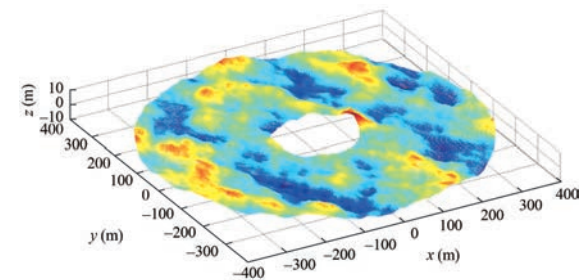
$$u_r' = u_r \times (1 + \varepsilon'), \varepsilon' \in (-\varepsilon, +\varepsilon) \quad (26)$$

where u_r' represents the velocity with error and ε' represents the relative error of velocity at a certain point, which is a random number within the selected error range $(-\varepsilon, +\varepsilon)$.

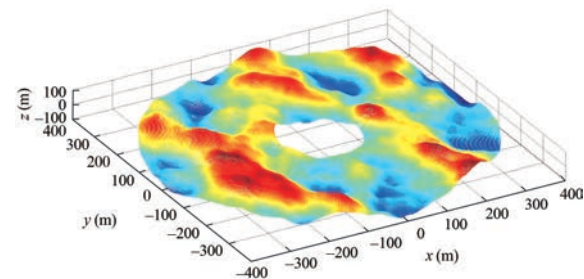
Figures 7(a) and 7(c) show the original and reconstructed velocity potentials, respectively, when the error of measurement velocity is 0.1%. The reconstructed velocity potential is accurate, and the figures of the original and inversion velocity potentials are almost the same. The comparison of the theoretical and reconstructed waves in Figures 7(b) and 7(d), respectively, is highly consistent.



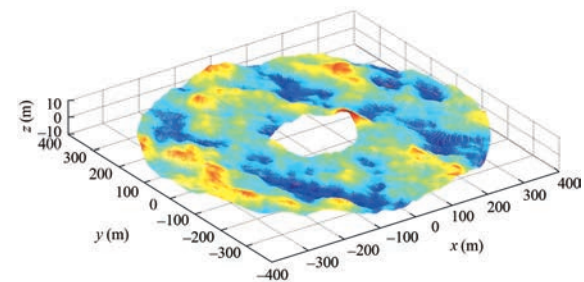
(a) Original velocity potential



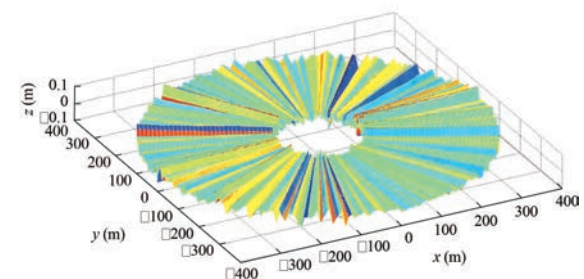
(b) Original wave surface



(c) Reconstructed velocity potential



(d) Reconstructed wave surface



(e) Reconstruction error

Figure 7 Reconstruction result with 0.1% velocity measurement error

Figure 7(e) shows that the absolute error of the reconstructed wave is controlled within 0.2 m, and the root mean square error is only 0.038 m, accounting for 0.5% of the significant wave height (7.5 m). The reconstructed result of the wave surface has high accuracy under this velocity error, which can meet the accuracy requirements of engineering.

When the error of the measured velocity is 0.5%, Figures 8(a) and 8(c) show that the reconstructed velocity potential is accurate, and the graph of the original velocity potential is more consistent with that of the reconstructed velocity potential. However, the result of the reconstructed wave surface (Figure 8(d)) is different from that of the original result (Figure 8(b)), which is manifested by some “jitters” in the adjacent radial direction. Compared with the reconstructed result of 0.1% velocity error, the wave surface becomes rough.

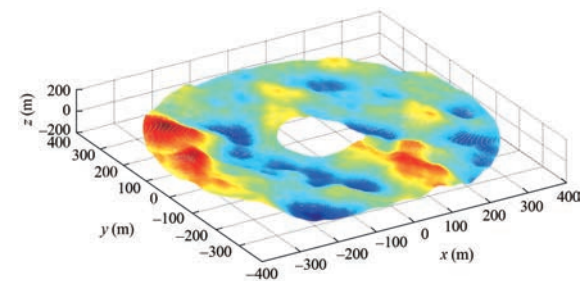
Figure 8(e) shows the absolute error chart of the wave surface. The figure reveals that the absolute error is controlled within 1 m. The root mean square error is only 0.218 m, accounting for 2.9% of the significant wave height (7.5 m), which indicates that highly accurate reconstructed results can still be obtained in most areas.

Figures 9(a)–9(d) show the theoretical velocity potential, reconstructed velocity potential, theoretical wave surface, and reconstructed wave surface when the error of measurement velocity is 1.0%. The wave surface and velocity potential are consistent with the theoretical velocity potential, and the original velocity potential is consistent with the inversion velocity potential.

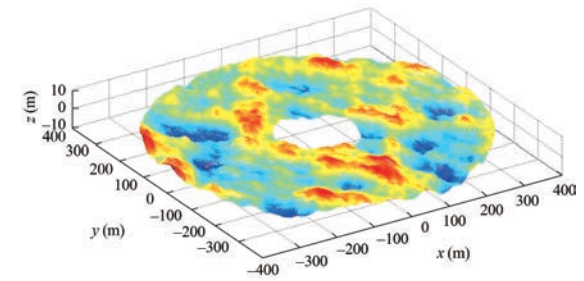
The chart of wave surface inversion reveals that “jitters” still appear in the reconstructed result (Figure 9(d)) compared with the original wave surface (Figure 9(b)), and the roughness of the entire wave surface is further increased compared with the reconstructed result of 0.1% and 0.5% velocity errors. However, the root mean square error is 0.433 m, accounting for 5.8% of the significant wave height (7.5 m), which is still in a low range, and the reconstructed results remain acceptable.

The comparison between Figures 10(a) and 10(c) shows that the accuracy of the reconstructed velocity potential is further reduced when the error of the measured velocity is 2%.

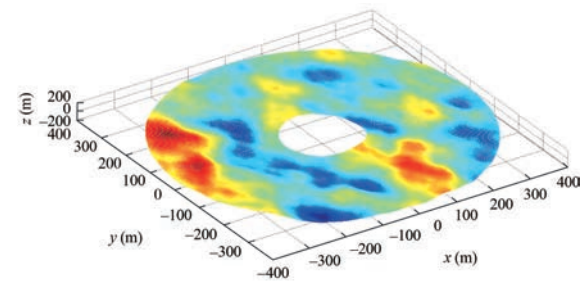
In terms of wave reconstruction, the reconstructed results (Figure 10(d)) are in good agreement with the theoretical results (Figure 10(b)). As shown in Figure 10(e), the root mean square error of wave reconstruction is 0.857 m, accounting for 11.4% of the significant wave height (7.5 m). However, when the boundary is forced, the artificial boundary will distribute its influence to the global when solving the reconstruction matrix, resulting in the deviation between the reconstructed and theoretical wave surfaces in the local region, and this deviation is consistent in the same radial direction. The maximum absolute error of the wave surface is close to 1.5 m, and the relative error is more than 20%.



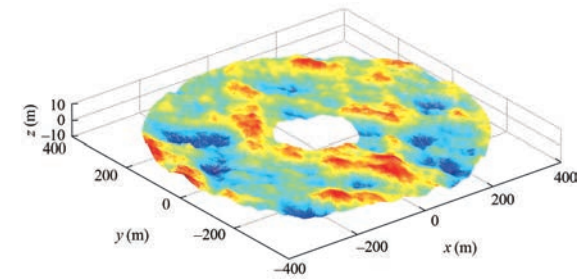
(a) Original velocity potential



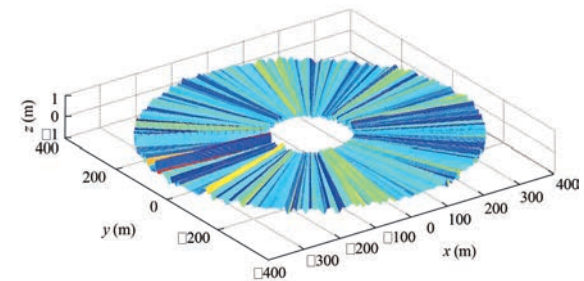
(b) Original wave surface



(c) Reconstructed velocity potential

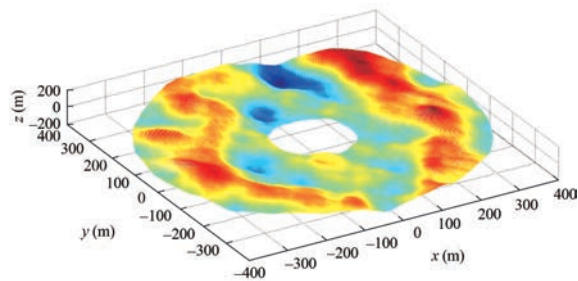


(d) Reconstructed wave surface

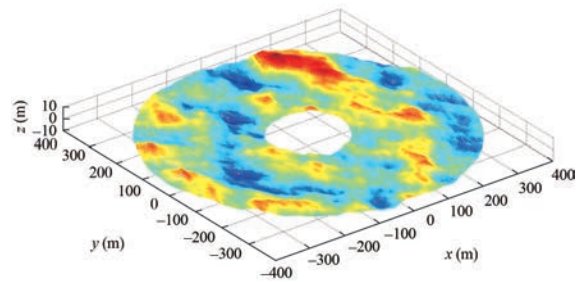


(e) Reconstruction error

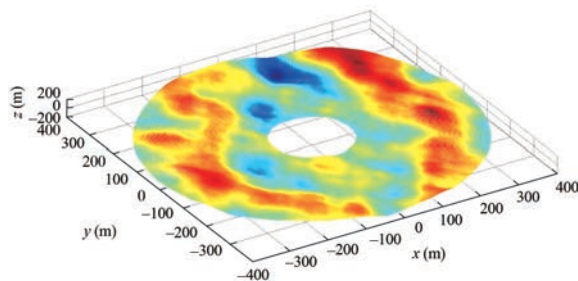
Figure 8 Reconstruction result with 0.5% velocity measurement error



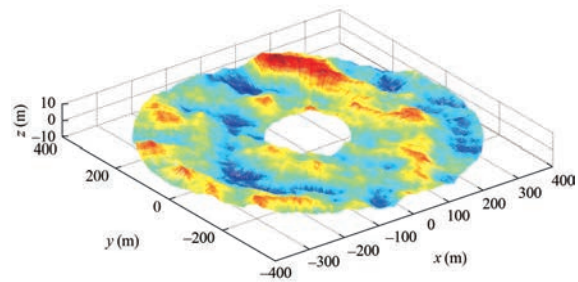
(a) Original velocity potential



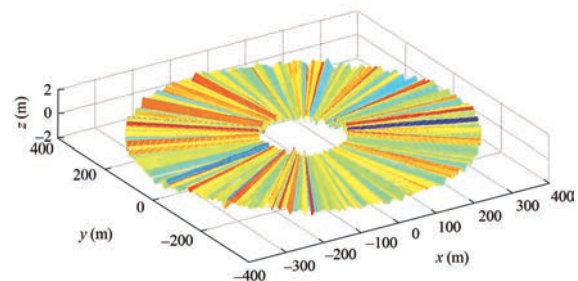
(b) Original wave surface



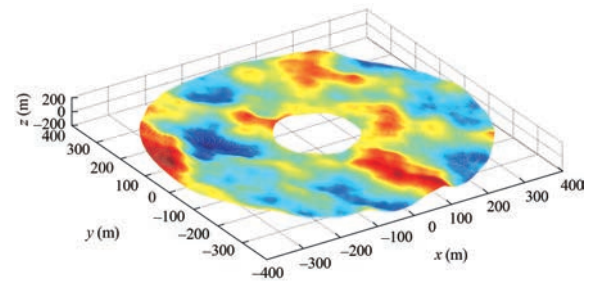
(c) Reconstructed velocity potential



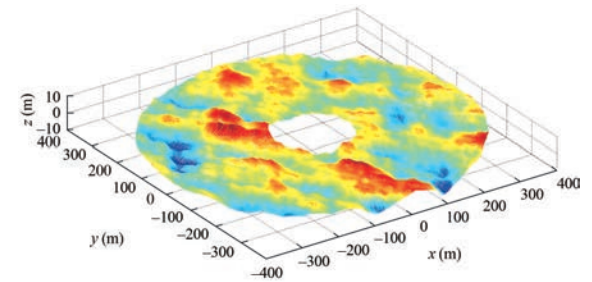
(d) Reconstructed wave surface



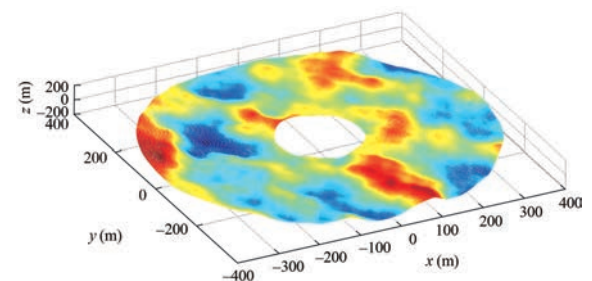
(e) Reconstruction error

Figure 9 Reconstruction result with 1.0% velocity measurement error

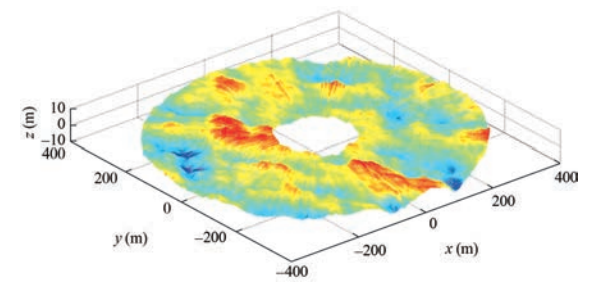
(a) Original velocity potential



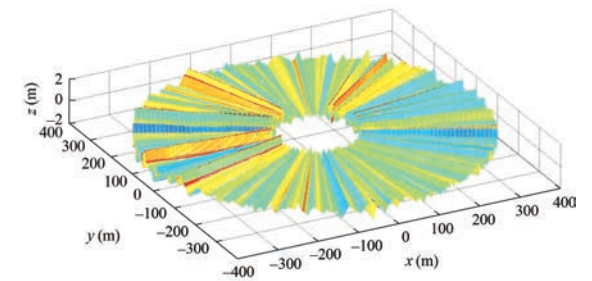
(b) Original wave surface



(c) Reconstructed velocity potential



(d) Reconstructed wave surface



(e) Reconstruction error

Figure 10 Reconstruction result with 2.0% velocity measurement error

This finding also reflects that the difference matrix method has high requirements for the accuracy of the measurement speed, and the increase of the measurement error will lead to a rise in the wave surface inversion error.

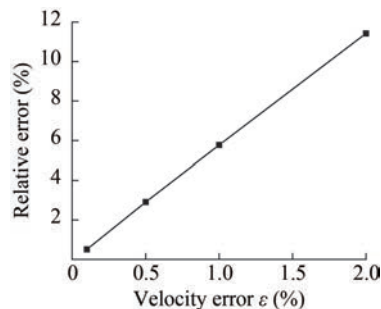


Figure 11 Reconstructed error with different velocity errors

6 Conclusions

In this paper, a study on the accurate inversion of wave field elevation based on coherent radar images is conducted. The traditional method of wave surface reconstruction using coherent radar involves the integration of the radial velocity. However, in the actual operation process, determining the initial value of the integral is difficult when obtaining the velocity potential. Therefore, this paper proposes a novel inversion approach that addresses the issue of initial values by employing a difference matrix method that does not require their determination. The inversion is performed through the difference of the velocity potential using the difference matrix approach.

The numerical simulation data of regular and short-crest waves are used to verify the model, and stability analysis with different velocity measurement errors is discussed. The conclusions are presented as follows:

1) For regular waves, when the wave heights are 3.5, 5.5 and 7.5 m, the reconstructed results have good agreement with the theoretical results, and the absolute errors are between 10^{-4} and 10^{-5} m, which verifies the feasibility of the inversion method.

2) In the context of short-crest waves, the performance of the inversion algorithm is evaluated for three sea states ranging from levels 5 to 7. The results show that the algorithm achieves high precision in inverting the irregular wave surface and the corresponding velocity potential under these representative sea conditions. The absolute errors of the wave surface are consistently within the range of 10^{-4} – 10^{-5} m, providing compelling evidence for the accuracy of the inversion algorithm.

3) Considering the influence of the measurement error of velocity, four measurement error values of 0.1%, 0.5%, 1.0%, and 2.0% are selected to analyze the stability of the reconstruction algorithm. When the measurement error increases, the wave surface reconstruction accuracy decreases,

and the root mean square error increases with the measurement error. The velocity potential reconstructed result is insensitive to the measurement error and is consistent with the original velocity potential image. The reconstructed results of wave surface and velocity potential of the reconstruction algorithm in the case of four kinds of measurement errors are generally consistent, and the reconstructed results are satisfactory.

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Competing interest The authors have no competing interests to declare that are relevant to the content of this article.

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