

Theoretical Investigation of Spherical Bubble Dynamics in High Mach Number Regimes

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Abstract

The compressibility of fluids has a profound influence on oscillating bubble dynamics, as characterized by the Mach number. However, current theoretical frameworks for bubbles, whether at the first or second order of the Mach number, are primarily confined to scenarios characterized by weak compressibility. Thus, a critical need to elucidate the precise range of applicability for both first- and second-order bubble theories arises. Herein, we investigate the suitability and constraints of bubble theories with different orders through a comparative analysis involving experimental data and numerical simulations. The focal point of our investigation encompasses theories such as the Rayleigh–Plesset, Keller, Herring, and second-order bubble equations. Furthermore, the impact of parameters inherent in the second-order equations is examined. For spherical oscillating bubble dynamics in a free field, our findings reveal that the first- and second-order bubble theories are applicable when $Ma \leq 0.3$ and 0.4 , respectively. For a single sonoluminescence bubble, we define an instantaneous Mach number, Ma_i . The second-order theory shows abnormal sensibility when Ma_i is high, which is negligible when $Ma_i \leq 0.4$. The results of this study can serve as a valuable reference for studying compressible bubble dynamics.

Keywords Bubble dynamics; Spherical bubble; Cavitation; Compressibility; Mach number

1 Introduction

Bubbles are widely applied in life and industry. They play a crucial role in diverse fields, including naval warfare (Hyunwoo et al., 2022; Li et al., 2014; Lu and Brown, 2021; Zhang et al., 2012; Zhang et al., 2011), seabed resource exploration using airgun bubbles (De Graaf et al., 2014; Li et al., 2020; Zhang et al., 2018), ultrasonic cleaning with microbubbles (Mason, 2016; Tuziuti, 2016), ice-breaking with oscillating bubbles (Cui et al., 2018), emulsification and food industry with bubbles in binary immiscible fluids (Li et al., 2023), and advanced medical

treatments (Akbar et al., 2021; Johnson, 1994; Kunkle and Beckman, 1983; Ohl et al., 2009). Nonetheless, the studies on oscillating bubbles characterized by high Mach numbers remain relatively scarce. Examples of such bubbles include those generated by deep-water explosions, where fluid compressibility becomes a critical factor needing special attention.

In the realm of bubble dynamics, theories with different orders of Mach numbers have been developed, offering convenient analytical tools. Pioneering work in bubble dynamics theory began with the Rayleigh–Plesset (RP) equation (Plesset, 1949; Lord Rayleigh, 1917), which can predict a spherical bubble oscillation based on the assumption of incompressibility. However, being a zeroth-order Mach number model, the RP equation implies undamped bubble oscillations and neglects energy radiation. Based on this, Herring (1941) introduced a bubble motion equation incorporating a first-order correction to account for the compressibility of the surrounding liquid. Keller and Kolodner (1956) proposed a compressible bubble theory by utilizing the wave equation rather than the Laplace equation. Building upon these advancements, Prosperetti and Lezzi (1986) derived a family of equations that described bubble wall motion with a first-order Mach number correction, encompassing both Herring’s and Keller’s formulations by modifying the parameter λ . Considering the essential

Article Highlights

- The suitability and constraints of bubble theories with different orders is investigated.
- The single sonoluminescence bubble dynamics is studied theoretically with the instantaneous Mach number.
- The bubble radius, bubble surface velocity and the dynamic pressure of the surrounding flow field is studied numerically and theoretically.

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effect of buoyancy and other factors (Zhang et al., 2015; Zhang and Ni, 2013), Zhang et al. (2023a; 2023b) proposed the unified bubble theory, considering various factors such as boundaries, viscosity, bubble interaction, compressibility, buoyancy, and so on, thereby contributing to a deeper understanding of bubble dynamics.

Regarding the second-order bubble theory, Tomita and Shima (1977) derived the second-order bubble motion equation by using the Poincaré–Lighthill–Kuo method to predict the spherical bubble dynamics in a viscous compressible fluid. Subsequently, Fujikawa and Akamatsu (1980) proposed a mathematical formulation accounting for the compressibility of the fluid, non-equilibrium condensation of the vapor, and so on. In addition, Lezzi and Prosperetti (1987) proposed a second-order bubble equation family by using the matched asymptotic expansions method.

There exist limitations to the application of the mentioned bubble theories when the Mach number of the oscillating bubble is relatively high. Moreover, the deviation between the bubble theories with different orders of Mach numbers must be studied further. Moreover, the results obtained from theories of the same Mach number order exhibit deviations due to the diverse mathematical formulations of bubble equations.

In this paper, we examine the applicability of bubble theories with different orders of Mach numbers through a comparative analysis involving experimental data and compressible numerical simulations. This paper is organized as follows. In Section 2, we introduce the bubble theories and numerical methods, providing a verification analysis of our numerical simulations. In Section 3, we study the spherical bubble dynamics by using the Keller equation with the form of enthalpy, second-order bubble theory, and numerical method to examine the applicability of bubble theories for different Mach numbers. In Section 4, we study single sonoluminescence bubble dynamics utilizing bubble theories with different orders of Mach numbers. Finally, Section 5 offers a concise summary of our findings.

2 Theory and numerical methods

2.1 Spherical bubble theories

With the assumption of an ideal and incompressible fluid, spherical bubble dynamics is always studied using the RP equation (Rozhdestvensky, 2022; Zhang and Ni, 2013). To study the bubble dynamics in the fluid considering compressibility, surface tension, and viscosity, the extended RP equations are proposed. Some essential assumptions are employed for a simpler deduction as follows: 1) Gravity is neglected. 2) Flow is irrotational. 3) Mass transfer or thermal effects during the bubble oscillation are ignored. 4) Bubble gas pressure is uniform. Then, the liquid is governed

by the equations of continuity and momentum as follows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \frac{\partial \mathbf{u}}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0 \quad (2)$$

where $\mathbf{u} = u\mathbf{r}$ is the velocity field, and ρ and p denote the density and pressure of the liquid, respectively. In an irrotational flow field, $\nabla \varphi = \mathbf{u}$, wherein φ denotes the velocity potential. Then, the equations of continuity and momentum become

$$\frac{1}{c^2} \left(\frac{\partial h}{\partial t} + \frac{\partial \varphi}{\partial r} \frac{\partial h}{\partial r} \right) + \nabla^2 \varphi = 0 \quad (3)$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} \left(\frac{\partial \varphi}{\partial r} \right)^2 + h = 0 \quad (4)$$

where the sound velocity c and enthalpy h in the liquid are defined as

$$c = \sqrt{\frac{dp}{d\rho}} \quad (5)$$

$$h = \int_{p_\infty}^p \frac{1}{\rho} dp \quad (6)$$

At the bubble surface, where $r=R(t)$, the kinematic boundary condition is

$$u(r=R(t), t) = \frac{dR}{dt} = \nabla \varphi \quad (7)$$

The equation of state (EOS) of the Tait form (Cole, 1948) is used to describe the water medium

$$\left(\frac{\rho_l}{\rho_\infty} \right)^n = \frac{p_l + B}{p_\infty + B} \quad (8)$$

Based on this, the explicit expressions of c and h become

$$c^2 = \frac{n(p_l + B)}{\rho_l} = c_\infty^2 + (n-1)h \quad (9)$$

$$h = \frac{c^2 - c_\infty^2}{n-1} = \frac{c_\infty^2}{n-1} \left[\left(\frac{p_l + B}{p_\infty + B} \right)^{\frac{n-1}{n}} - 1 \right] \quad (10)$$

where the empirical constants $B=3\,049$ bar and $n=7.15$ for water. c_∞ denotes the speed of sound in the liquid at infinity, which can be obtained by

$$c_\infty^2 = \frac{n(p_\infty + B)}{\rho_\infty} \quad (11)$$

The equations mentioned above can be closed with

$$p_b = p_g - \frac{2\sigma}{R} - \frac{4\mu\dot{R}}{R} \quad (12)$$

where

$$p_g = p_0 \left(\frac{R_0}{R} \right)^{3\gamma} + p_v \quad (13)$$

Here, p_g refers to the bubble gas pressure, and the vapor pressure $p_v = 2338$ Pa. σ and μ denote the surface tension and viscosity coefficients of the liquid, and the polytropic exponent γ is set as 1.25. We ignore the saturated vapor pressure, considering the considerably high non-condensable gas pressure. p_0 is the initial bubble pressure, and we define $\varepsilon = p_0/p_\infty$ as the strength parameter. In this paper, the Mach number (Fuster and Popinet, 2018; Li et al., 2021) is defined as

$$Ma = \frac{\sqrt{\frac{p_0 - p_\infty}{\rho_\infty}}}{c_\infty} \quad (14)$$

where the numerator $\sqrt{(p_0 - p_\infty)/\rho_\infty}$ almost represents the average velocity of bubble oscillation. In this paper, we focus on the bubble dynamics at different Mach numbers; thus, the ambient pressure p_∞ and the initial bubble gas

pressure p_0 can be obtained from Eq. (14) with known Ma and ε :

$$p_\infty = \frac{Ma^2 n B}{\varepsilon - 1 - Ma^2 n} \quad (15)$$

$$p_0 = \varepsilon p_\infty = \frac{\varepsilon Ma^2 n B}{\varepsilon - 1 - Ma^2 n} \quad (16)$$

Based on the simplified matched asymptotic expansions method (Lagerstrom and Casten, 1972), Prosperetti and Lezzi (1986) presented a bubble wall motion equation family with the first order of the Mach number:

$$\left[1 - (\lambda + 1) \frac{\dot{R}}{c_\infty} \right] R \ddot{R} = \frac{3}{2} \left[1 - \frac{1}{3} (3\lambda + 1) \frac{\dot{R}}{c_\infty} \right] \dot{R}^2 = \left[1 + (1 - \lambda) \frac{\dot{R}}{c_\infty} \right] \left(h_b - \frac{p_v}{\rho_\infty} \right) + \frac{R}{c_\infty} \frac{d}{dt} \left(h_b - \frac{p_v}{\rho_\infty} \right) \quad (17)$$

where h_b denotes the enthalpy at the bubble surface, and p_v refers to the vapor pressure. When $\lambda=0$ and 1, Eq. (17) reduces to the Keller and Herring equations in the form of enthalpy, respectively. We will compare the equations with different λ values in the later section.

Based on the rigorously matched asymptotic expansions method, the second-order bubble motion equation family (Lezzi and Prosperetti, 1987) can be obtained as follows:

$$\left[1 - (\lambda + 1) \frac{\dot{R}}{c_\infty} + \left(\frac{14}{5} + 2\lambda + \theta \right) \frac{\dot{R}^2}{c_\infty^2} \right] R \ddot{R} + \frac{3}{2} \left[1 - \left(\lambda + \frac{1}{3} \right) \frac{\dot{R}}{c_\infty} + \left(\frac{16}{15} + \frac{4}{3} \lambda + \theta \right) \frac{\dot{R}^2}{c_\infty^2} \right] \dot{R}^2 + \frac{R^2 \ddot{R}^2}{c_\infty^2} = \left[1 + (1 - \lambda) \frac{\dot{R}}{c_\infty} + \theta \frac{\dot{R}^2}{c_\infty^2} \right] h_b + \left[1 - (1 + \lambda) \frac{\dot{R}}{c_\infty} \right] \frac{\dot{R}}{c_\infty} \dot{h}_b \quad (18)$$

where λ and θ are arbitrary parameters. More details about the derivation can be found in the reference (Lezzi and Prosperetti, 1987; Prosperetti and Lezzi, 1986).

Regarding a spherical bubble oscillation, the surrounding flow field is symmetric about the bubble center. The bubble oscillation will disturb the surrounding flow field, and the pressure of the liquid can be obtained from

$$p_l = \rho \left(\frac{2R\dot{R}^2 + R^2\ddot{R}}{r} - \frac{1}{2} \frac{R^4\dot{R}^2}{r^4} \right) \quad (19)$$

where r is the distance between the bubble center and the measurement point.

2.2 Numerical method

We use the multiphase solver *compressibleInterFoam* implemented in OpenFOAM to predict the bubble dynamics in a compressible free field. The equations are discretized and solved using the finite volume method (FVM). The volume-of-fluid method is used to capture the gas-liquid

interface and simulate the bubble oscillation. Notably, the bubble dynamics in a free field can be reduced to an axisymmetric problem to improve the computational efficiency. In this paper, the computational domain is shaped like a wedge, as shown in Figure 1. Regarding the boundary conditions on the right-hand side, top, and bottom of the domain, we employ *zeroGradient* and *pressureInletOutletVelocity* for pressure field p and velocity field U , respectively. The front and back planes of the wedge-shaped domain are applied with the wedge boundary condition. The left boundary of the wedge forms the axis. In this paper, the domain boundaries are placed around 200 times the maximum bubble radius away from the bubble center to prevent the influence of the reflected wave.

The heat and mass transfer, as well as the phase change between the gas and liquid phases, are negligible, then the governing equations become

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (20)$$

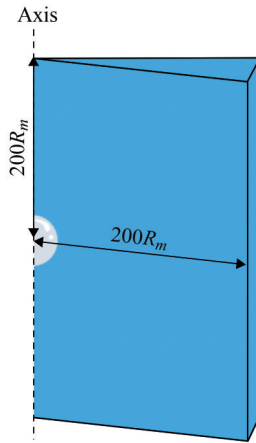


Figure 1 Computational domain for the numerical simulation

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla p + \nabla \cdot \mathbf{s} + \mathbf{f} \quad (21)$$

where $\mathbf{u}$ refers to the fluid velocity vector, $\mathbf{s}$ and $\mathbf{f}$ are the viscous stress tensor and surface tension, respectively.

The surrounding liquid follows the Tait EOS mentioned before, and the bubble content is described by the polytropic EOS as follows:

$$\frac{p_b}{\rho_g^r} = \frac{p_\infty}{\rho_\infty^r} \quad (22)$$

where the reference density of the liquid $\rho_\infty = 998 \text{ kg/m}^3$, density of the bubble gas $\rho_g = 1.29 \text{ kg/m}^3$, and specific heat ratio $r = 1.25$. During the calculation, the maximum Courant number is set as 0.1 to maintain stability.

2.3 Verification analysis

Before the series of simulations at different Mach numbers, we verify the numerical results by studying a low Mach number case ($Ma = 0.03$) with both theoretical and numerical methods. When the Mach number is sufficiently low, the compressible bubble theories are reliable. We set $\lambda = 1$ and $\theta = 0$ for the second-order bubble equation, i.e., the Tomita and Fujikawa form (Fujikawa and Akamatsu, 1980; Tomita and Shima, 1977). The comparison of the second-order equations with different λ values will be shown later. The Keller form written in terms of enthalpy has been verified as the most accurate first-order equation, which is studied in the reference (Prosperetti and Lezzi, 1986) in detail. In this case, the initial bubble radius is 3 mm, ambient pressure p_∞ is 217 720 Pa, and the strength parameter is set as $p_0 = 10p_\infty$.

Figure 2 illustrates the temporal evolution of the bubble radius during the two initial periods of oscillation. LPE denotes the second-order Prosperetti & Lezzi equation. The remarkable agreement between theoretical predictions and numerical simulations proves the precision of the numer-

ical method. The bubble radius reaches 0.008 m in the first oscillation period. Due to energy dissipation, the maximum bubble radius during the second period R_{m2} is approximately 4.4% smaller than R_{m1} . Figure 3 shows the dynamic pressure variation in the flow field at a fixed point $r = 0.015 \text{ m}$. The pressure peak is related to the minimum bubble radius, and it falls following the attenuation of the bubble oscillation. In conclusion, when the acoustic influence is relatively weak, our numerical results exhibit an exceptional degree of concurrence with both first- and second-order theoretical predictions for a spherical oscillating bubble. This congruence underscores the reliability and credibility of our numerical model.

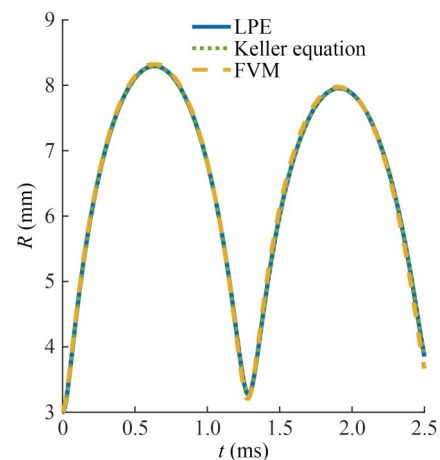


Figure 2 Bubble radius when the Mach number is 0.03

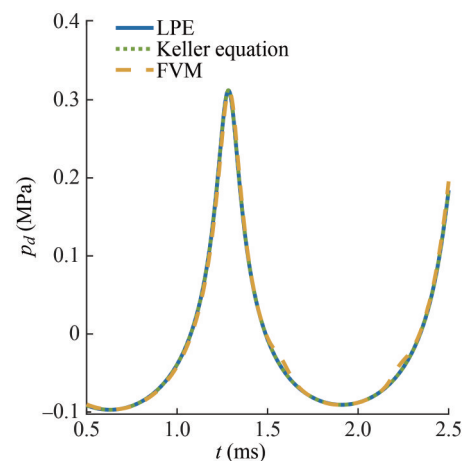


Figure 3 Dynamic pressure of $r = 5R_0$ in the fluid field

3 Spherical bubble dynamics

It is efficient and convenient to study the spherical bubble dynamics using bubble theories. However, as weakly compressible methods, the limitations emerge in their applicability when confronted with relatively high Mach

number cases. The objective is to delineate the boundaries of their utility through a rigorous comparison between theoretical predictions and numerical simulations.

The second-order bubble wall equation family (Lezzi and Prosperetti, 1987) encompasses a range of equations distinguished by varying parameters. We conduct a comparative analysis, pitting the outcomes derived from the equations featuring distinct parameter sets (Fujikawa and Akamatsu, 1980; Tomita and Shima, 1977; Wang et al., 2021) against those obtained through numerical methodologies. Specifically, we fix the Mach number at 0.6 for this investigation. As shown in Figure 4, the results reveal a degree of deviation, particularly evident at the maximum bubble radius. Notably, among the considered parameter sets, the theoretical result with $\lambda=1$ is the closest to the numerical result (see inset). Thus, we employ the second-order equation with $\lambda=1$ (Tomita and Fujikawa form) in this section.

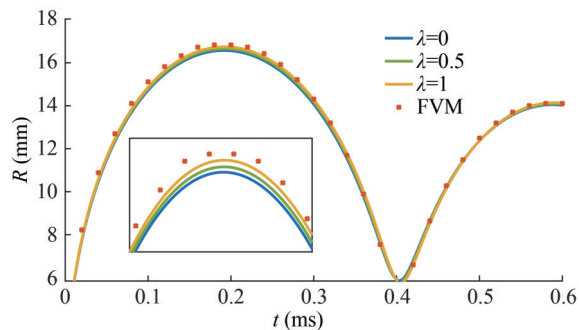


Figure 4 Comparison of bubble radius curves between the numerical simulation and second-order equations with $\lambda = 0, 0.5$, and 1. The Mach number of this case is set as 0.6

We study the cases with $Ma=0.1-0.6$ using the first- and second-order theories and the numerical method. The initial bubble radius is set as 3 mm, and $p_0=100p_\infty$ in all the cases. Figures 5–7 demonstrate the comparison of various bubble dynamics between the numerical and theoretical results. In this section, the Keller equation refers to the equation written in terms of enthalpy. Accordingly, Tables 1–3 present the bubble dynamic behavior at different Mach numbers, demonstrating deviations between the theoretical and numerical results, where ε_1 and ε_2 denote the deviation between the theoretical result with different orders of Mach numbers and the numerical results. As shown in Figure 5, when $Ma \leq 0.4$, the second-order theoretical results closely match the numerical ones, and the deviation is under 0.3%. This suggests that the second-order bubble theory can predict the spherical bubble oscillation precisely when the Mach number is under 0.4. The difference between the second-order theory and numerical results gradually becomes larger when $Ma > 0.4$, reaching about 0.8% and 1% at $Ma=0.5$ and 0.6, respectively. The first-order theory provides reliable results in a more limited range due to its

relatively lower accuracy. The deviation becomes 0.8% when $Ma=0.3$ and reaches up to 3.2% and 4.3% when $Ma=0.5$ and 0.6, respectively.

Table 1 Maximum bubble radius R_m (mm) at different Ma values

Ma	Keller	LPE	FVM	ε_1 (%)	ε_2 (%)
0.1	19.3	19.3	19.3	0	0
0.2	18.55	18.59	18.6	0.3	0.1
0.3	17.9	18	18.04	0.8	0.2
0.4	17.26	17.53	17.58	1.8	0.3
0.5	16.66	17.07	17.21	3.2	0.8
0.6	16.15	16.7	16.87	4.3	1

Table 2 Maximum bubble surface velocity U_m (m/s) at different Ma values

Ma	Keller	LPE	FVM	ε_1 (%)	ε_2 (%)
0.1	65.73	65.73	65.73	0	0
0.2	125.4	125.7	125.8	0.3	0.1
0.3	179.3	181	183	2	1.1
0.4	227.5	232.1	238	4.4	2.5
0.5	270.7	280.5	296	8.5	5.2
0.6	313.2	327	348	10	6

Table 3 Maximum dynamic pressure p_d (MPa) in the fluid field at $r=5R_0$ at different Ma values

Ma	Keller	LPE	FVM	ε_1 (%)	ε_2 (%)
0.1	2.74	2.74	2.74	0	0
0.2	7.56	7.52	7.44	1.6	1.1
0.3	12.75	12.98	13.15	3	1.3
0.4	17.7	18.64	19	6.8	1.9
0.5	22.66	24.51	25.59	11.4	4.2
0.6	27.65	31.32	32.77	15.6	4.4

As shown in Figure 6, we can find similar phenomena. For the bubble oscillating velocity, the first- and second-order results gradually deviate from the numerical one as the Mach number increases. The second-order results coincide well with the numerical one when $Ma \leq 0.4$, with a deviation under 2.5%. When $Ma=0.6$, the deviation reaches 6%. For the Keller equation, when $Ma \leq 0.3$, the deviation is under 2%, which reaches 4.4% and 10% at $Ma=0.4$ and 0.6, respectively.

Subsequently, we examine the perturbed flow field surrounding the oscillating bubble. The probe is set at $r=5R_0$ to detect the dynamic pressure, and the initial conditions

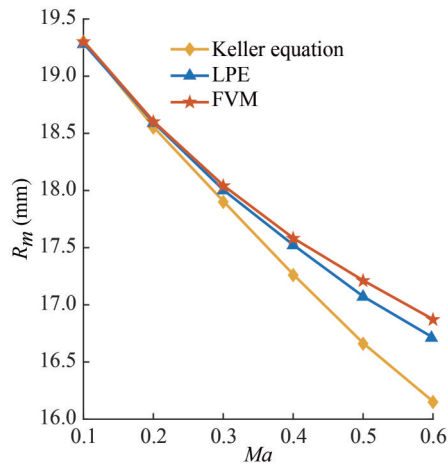


Figure 5 Maximum bubble radius at different Mach numbers

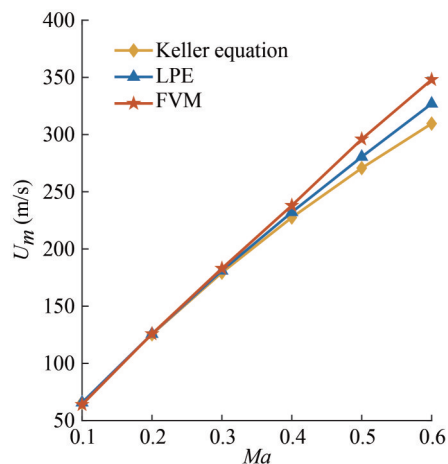


Figure 6 Maximum bubble surface velocity at different Mach numbers

remain consistent with those previously described. Figure 7 demonstrates the pressure peak in the flow field obtained theoretically and numerically at different Mach numbers. Eq. (19) suggests that the bubble surface velocity plays a vital role in the pressure field, and the bubble oscillating velocity increases dramatically with the Mach number. Thus, the maximum pressure increases with the Mach number due to the higher oscillating velocity despite the smaller bubble volume. When $Ma \leq 0.4$, the second-order theory and numerical method yield extremely close results, and the deviation is under 2%. When $Ma > 0.4$, discrepancies become increasingly noticeable, eventually reaching 4.4% at $Ma = 0.6$. For the first-order result, it coincides well with the numerical result when $Ma \leq 0.3$, and the deviation is under 2%. At $Ma = 0.6$, the deviation becomes about 16%.

Based on the preceding discussion, we can ascertain the suitability of both first and second-order bubble theories when investigating the dynamics of spherical bubbles within a compressible free flow field. We set 3% as the judgment criterion of the deviation; thus, the first-order theory can simulate the spherical bubble precisely when $Ma \leq 0.3$,

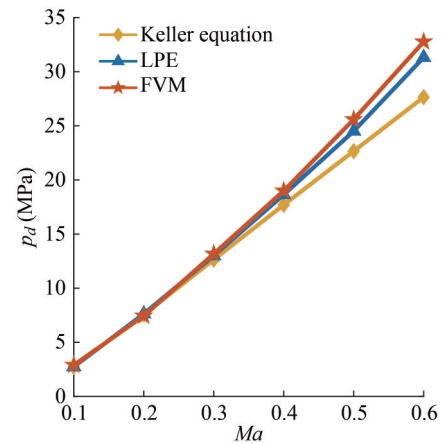


Figure 7 Maximum dynamic pressure in the fluid field at $r=5R_0$ at different Mach numbers

while the second-order theory is applicable when $Ma \leq 0.4$. However, notably, current bubble theories applicable most of the time because high Mach number bubble pulsations are relatively uncommon.

Then, we compare the first- and second-order bubble theories more intuitively. Figures 8 and 9 illustrate the temporal evolution of the oscillating bubble radius calculated by the first- and second-order theories for $Ma = 0.1$ and 0.6. Notably, the pulsation amplitude reduces clearly as the Mach number increases due to higher energy dissipation. In addition, the results of the first- and second-order theories coincide perfectly when $Ma = 0.1$, and the first-order result deviates from the second-order one when $Ma = 0.6$. In an oscillating period, the deviation becomes larger as the bubble expands and becomes smaller when the bubble collapses. Moreover, the deviation generally decreases along the bubble oscillation. The maximum deviations in the first and third periods are 3.6% and 1.6%, respectively; thus, the results coincide better in latter periods than in the first period.

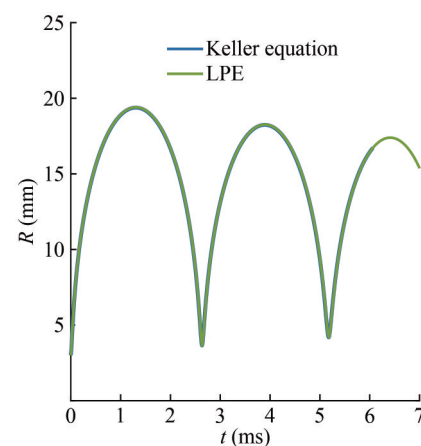


Figure 8 Comparison between first- and second-order bubble radii at $Ma = 0.1$

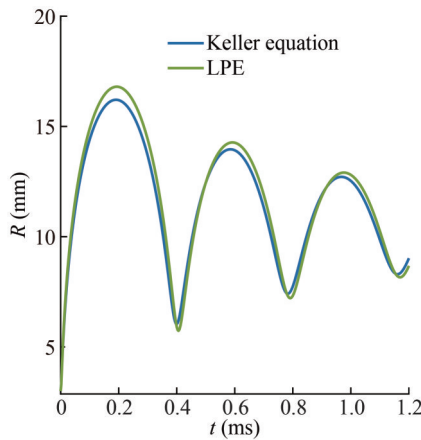


Figure 9 Comparison between first- and second-order bubble radii at $Ma = 0.6$

4 Sonoluminescence cavitation bubbles

In addition to the bubble oscillating due to considerable pressure difference, another type of cavitation bubble oscillates due to acoustic wave stimulation. Notably, the Mach number reaches exceptionally high levels instantaneously during the rebounding phase of a sonoluminescence bubble. In this section, we study the sonoluminescence cavitation bubbles to compare the capability of different bubble theories of the zeroth-, first- and second-order. In addition, for the second-order bubble theory, different λ values lead to different results; thus, we compare the results of different constant parameter λ values with the experimental data to choose a more reliable form. Moreover, regarding the first-order theories, discrepancies emerge between the predictions of the Keller and Herring equations, necessitating a comparative evaluation.

We study a single sonoluminescence cavitation bubble using the RP equation, Keller equation, Herring equation, and the second-order bubble theories with $\lambda=1$ and 0. The initial bubble radius is set at $4.7 \mu\text{m}$, and the initial bubble gas pressure is equal to the pressure at the bubble surface, as shown below:

$$p_0 = p_\infty + \frac{2\sigma}{R} + \frac{4\mu\dot{R}}{R} \quad (23)$$

where the surface tension and fluid viscosity are set as $\sigma=0.075 \text{ N/m}$ and $0.001 \text{ Pa}\cdot\text{s}$, respectively. The bubble is stimulated by the high-frequency acoustic wave set from $t=0$:

$$p_{\text{acoustic}} = -P_a \sin(2\pi ft) \quad (24)$$

where $P_a=136 \text{ kPa}$ and $f=29 \text{ kHz}$. The bubble gas pressure can be obtained by

$$p_g = p_0 \left(\frac{R_0}{R} \right)^{3\gamma} + p_v \quad (25)$$

where the polytropic exponent is $\gamma=1.4$.

The comparison of the sonoluminescence bubble radii between the experimental data from Matula (1999) and the theoretical results are shown in Figure 10. The bubble radius expands to the maximum value of $40 \mu\text{m}$ at about $t=28 \mu\text{s}$ and then collapses. We can find that all theoretical results coincide well with the experimental result in the first oscillating period. The maximum bubble radius in the second period R_{m2} is about $11 \mu\text{m}$, 72% smaller than the first period due to the severe energy dissipation. RP equation does not consider the fluid compressibility; thus, the major part of energy dissipation is ignored, and the result deviates dramatically from the experimental data in the second period. Unexpectedly, the second-order theory shows great sensibility. When $\lambda=1$, it overestimates the bubble radius in the second oscillating period, with 71% higher R_{m2} compared with the experimental one. However, when $\lambda=0$, the second-order result coincides well with the experimental data. The sensibility of the second-order bubble theory will be discussed later. In addition, both the Keller and Herring results show good accuracy.

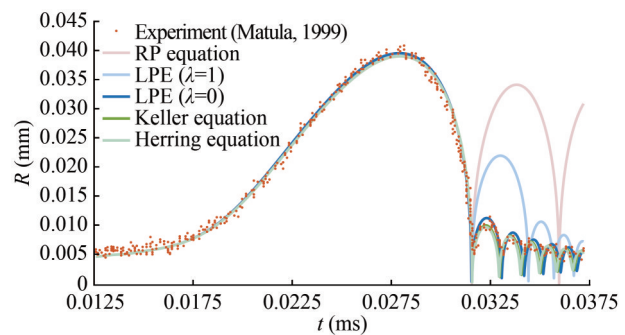


Figure 10 Comparison of the sonoluminescence bubble radii between the experiment and theories with different orders of Mach number

The sonoluminescence bubble is characterized by high oscillating velocity and frequency. Here, we define the instantaneous Mach number $Ma_i = U/c_\infty$ due to the changing ambient pressure; thus, we can focus on the fluid compressibility at every moment. Figure 11 demonstrates the bubble surface velocity obtained from the Keller equation, Herring equation, and second-order theory with different λ values. Evidently, the maximum bubble surface velocity reaches $O(10^3) \text{ m/s}$; thus, the maximum Ma_i increases dramatically. As discussed before, the second-order equation ($\lambda=0$), Keller equation, and Herring equation show good accuracy. However, the second-order equation, which is with a higher order of the Mach number, shows abnormal sensibility. This suggests that the present theoretical frame is not quite suitable in cases with high Ma_i . In addition, we find some deviations in the bubble velocities between the first and second-order results. The maximum bubble collapsing velocity obtained by the second-order theory is lower than that from the first-order theory, while the maximum rebounding velocity

is relatively higher. The second-order bubble rebounds slightly earlier than the first-order one, with a deviation of about 0.09%.

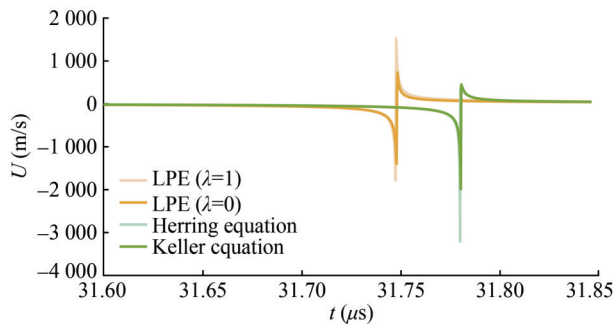


Figure 11 Comparison of the sonoluminescence bubble surface velocity between different bubble theories

To examine the sensibility of the second-order bubble theory, we compare the bubble dynamics between $\lambda=0$ and 1. The Mach number is set from 0.01 to 0.7, and other initial conditions remain the same as before. As shown in Figure 12, the relative deviation of the maximum bubble radius of the second period R_{m2} increases monotonically with the Mach number, i.e., below 3% when $Ma_i \leq 0.4$. The growth trend gradually becomes stronger when $Ma_i > 0.4$, and the relative deviation reaches about 22% at $Ma_i = 0.7$. Accordingly, the bubble radius curves at $Ma = 0.7$ are shown in Figure 13. The results coincide perfectly in the first period, while the deviations about both bubble radius and oscillating frequency emerge from the second period. The bubble oscillation when $\lambda = 1$ is always stronger than when $\lambda = 0$, and the same can be found in Figure 10.

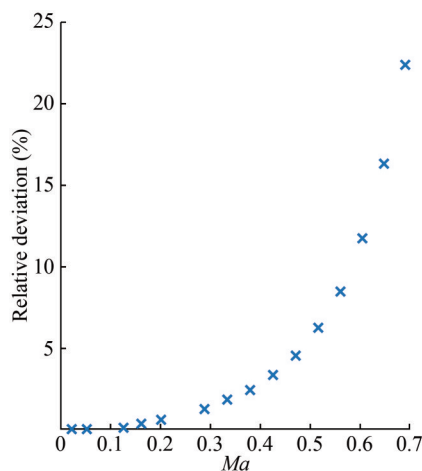


Figure 12 Relative deviation of R_{m2} with different Mach numbers

Notably, although the second-order equation ($\lambda=0$), Keller and Herring equations can predict the sonoluminescence bubble oscillation well at a high instantaneous Mach number Ma_i , the second-order theory shows abnor-

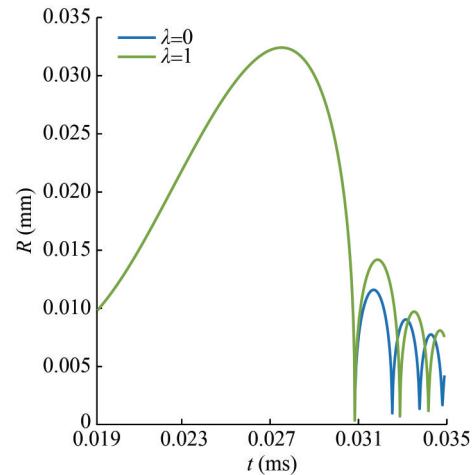


Figure 13 Bubble radius curves when $\lambda=0$ and 1

mal sensibility. Evidently, this type of sensibility is weak when $Ma \leq 0.4$. Overall, the sonoluminescence bubble dynamics must be studied theoretically when the maximum Ma_i is under 0.4, or the accuracy of the theory must be tested before studying a case at a high instantaneous Mach number.

5 Conclusions

In this study, we have examined the dynamics of spherical bubbles and a single sonoluminescence bubble by employing various bubble theories and numerical simulations. Our investigation has revolved around assessing the applicability and constraints of the first- and second-order bubble theories through comparisons with the FVM simulations and experimental data. The principal outcomes of our research are summarized as follows:

1) The numerical method is validated at a low Mach number. For a spherical bubble oscillating in a free field, the first-order bubble theory is applicable when $Ma \leq 0.3$, with the relative deviation from the numerical result under 3%. Beyond the threshold, the deviation increases monotonically with the Mach number. At $Ma = 0.6$, the deviation in the maximum bubble radius and velocity and the dynamic pressure peak reaches 4.3%, 10%, and 16%, respectively.

2) For a spherical bubble oscillating in a free field, the second-order bubble theory is applicable when $Ma \leq 0.4$, with a relative deviation under 3%. At $Ma = 0.6$, the deviation in the maximum bubble radius and velocity and the dynamic pressure peak reaches 1%, 6%, and 4.4%, respectively.

3) For a single sonoluminescence bubble, the first-order theoretical result coincides well with the experimental data when the instantaneous Mach number Ma_i is substantially high, while the second-order theory shows abnormal sensibility. It suggests that the present theoretical frame is not as suitable for high instantaneous Mach number cases. The sen-

sibility of the second-order bubble theory is negligible when $Ma_i \leq 0.4$; thus, the theoretical study is reliable when $Ma_i \leq 0.4$.

In practice, high Mach number ($Ma > 0.3$) bubble pulsations are relatively rare, manifesting primarily in extreme scenarios such as deep-sea detonations and phenomena such as sonoluminescence. Hence, contemporary first- and second-order bubble theories remain applicable in the majority of the cases.

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