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RESEARCH ARTICLE

Buckling Analysis of Tapered Continuous Columns by Using Modified Buckling Mode Shapes

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Abstract
Elastic critical buckling load of a column depends on various parameters, such as boundary conditions, material, and cross-section geometry. The main purpose of this work is to present a new method for investigating the buckling load of tapered columns subjected to axial force. The proposed method is based on modified buckling mode shape of tapered structure and perturbation theory. The mode shape of the damaged structure can be expressed as a linear combination of mode shapes of the intact structure. Variations in length in piecewise form can be positive or negative. The method can be used for single-span and continuous columns. Comparison of results with those of finite element and Timoshenko methods shows the high accuracy and efficiency of the proposed method for detecting buckling load.

Keywords Buckling analysis · Tapered column · Continuous columns · Finite element method · Modified buckling mode shapes

1 Introduction

Structures, such as beams and columns, are widely used as slender structures in engineering applications, including in bridges, aerospace, and marine industries. To maximize the load capacity of structures and minimize the total weight for the optimized design, scholars have proposed the use of non-uniform column with cross section varying along their length.

When a column is exposed to compressive load greater than its capacity, it deviates from the original stable equilibrium state and buckling occurs. Buckling loads must be investigated to avoid catastrophic failure in stepped columns. The presence of stepped parts depends on the proportion and location of variations and will result in changes in stress distribution within the member buckling load and mode shapes.

Studies about the stability of non-uniform columns have been published. Different methods have been used to calculate the buckling load of columns. Dinnik (1929) and Timoshenko and Gere (1961) found closed solutions for differential equation of buckling. Gere and Carter (1962) reached an exact solution for buckling of non-uniform columns with simply supported boundary condition by using Bessel's functions. O'Rourke and Zebrowski (1977) presented an approximated method based on finite difference method to calculate the buckling load of non-prismatic columns. Ermopoulos (1986) investigated the buckling load of non-uniform columns under axial concentrated forces. Smith presented an analytic solution for buckling of non-uniform members. Comparison of his results with those provided by Dinnik showed that, for small taper ratios, the error is small but increases as the taper becomes severe (Smith 1988). Williams and Aston (1989) developed some diagrams and used them to calculate the buckling load of tapered columns. Arbabi and Li (1991) presented a nonlinear method for buckling of elastic columns with gradually various thicknesses. Bazeos and Karabalis (2006) proposed an approximate method based on a series of dimensionless design-oriented charts relating the critical load of linearly tapered columns of I section to the taper

Article Highlights
• Buckling loads must be investigated to avoid catastrophic failure in tapered columns.
• Eigenbuckling equation and perturbation method are used to calculate the mode shapes of tapered structures.
• The proposed method is tested for buckling of continuous columns contrary to previous research.
• Several examples are presented, and results are compared with those of finite element and Timoshenko methods.

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ratio and boundary condition. Rahai and Kazemi (2008) formulated a procedure for buckling analysis of tapered column members by using modified vibrational mode shape and energy method.

Atay and Coşkun (2009) developed a variational iteration method to solve the stability problem of homogeneous Euler column with elastic restraint. Huang and Li (2012) proposed an analytic approach to determine critical buckling loads of non-uniform columns with elastic restraint along their length; they transform the problem into a Fredholm equation and then to a system of linear equations. Serna et al. (2011) presented a closed-form solution for buckling of uniform members under non-uniform axial load; they proposed an equivalent load approach for non-uniform members subjected to non-uniform axial load distribution. Yilmaz et al. (2013) introduced a localized differential quadrature method for buckling analysis of axially functionally graded non-uniform columns with elastic restraints. Rajasekaran (2013) used differential transformation (DT)–based dynamic stiffness approach to solve the buckling equation of axially functionally graded non-uniform beams. Afsharfard and Farshidianfar (2014) investigated the buckling load of non-uniform columns by using iteration perturbation method. Trahair (2014) presented a finite element method to calculate the buckling capacity of tapered beam structures under out-of-plane loads. Zhang et al. (2016) developed the Hencky bar-chain model for buckling and vibration analyses of non-uniform beams resting on partial variable elastic foundation.

In this work, a new method is presented to calculate buckling load by using modified mode shape in columns with cross section varying along their length. Variations in the length of the column are considered as piecewise. The proposed method can be used to calculate buckling load of continuous columns with variable cross section that is difficult to solve analytically.

Several numerical examples concerning stability behavior of tapered columns will be discussed to demonstrate the efficiency and accuracy of the developed approach. In these examples, the effects of boundary conditions, column length, and increase or decrease in stiffness variation were investigated.

2 Buckling Theory of Tapered Structure

For an intact structure, eigenbuckling equation is given by:

$$(\boldsymbol{K}-\boldsymbol{\lambda}_i\boldsymbol{K}_G)\varphi_i = 0 \qquad (1)$$

where $\boldsymbol{K}(n \times n)$ and $\boldsymbol{K}_G(n \times n)$ are stiffness and geometric stiffness matrices of the structure, respectively. $\boldsymbol{\lambda}_i$ and φ_i are the ith eigenvalue and buckling mode shapes of the structure, respectively. The minimum value of $\boldsymbol{\lambda}_i$ is known as buckling load. Buckling mode shape can be normalized as follows:

$$\varphi_i^{\mathrm{T}}\boldsymbol{K}_G\varphi_j^{\mathrm{T}} = \delta_{ij} \qquad (2)$$

δ_{ij} is the Kronecker's delta.

Changes in structures can cause changes in the stiffness matrix; as such, buckling load and mode shape will also change. These changes are introduced by $\delta\boldsymbol{\lambda}_i$ and $\delta\varphi_i$, respectively. Geometric stiffness will be changed, but it can be neglected.

For damaged structure, the eigenbuckling equation using perturbation method can be described as:

$$(\boldsymbol{K}+\delta\boldsymbol{K}-(\boldsymbol{\lambda}_i+\delta\boldsymbol{\lambda}_i)\boldsymbol{K}_G)(\varphi_i+\delta\varphi_i) = 0 \qquad (3)$$

Ignoring upper the differential terms in Eq. (3):

$$\boldsymbol{K}\delta\varphi_i-\boldsymbol{\lambda}_i\boldsymbol{K}_G\,\delta\varphi_i = -\delta\boldsymbol{K}\varphi_i+\delta\boldsymbol{\lambda}_i\boldsymbol{K}_G\varphi_i \qquad (4)$$

Premultiplying Eq. (4) by $\varphi_i^{\mathrm{T}}$and using Eq. (2) yields:

$$\varphi_i^{\mathrm{T}}(\boldsymbol{K}-\boldsymbol{\lambda}_i\boldsymbol{K}_G)\,\delta\varphi_i = -\varphi_i^{\mathrm{T}}\delta\boldsymbol{K}\varphi_i+\delta\boldsymbol{\lambda}_i \qquad (5)$$

2.1 First Order

In this study, the change in buckling load is presented in two orders. The change in mode shape is neglected in the first order and is assumed as a linear combination of the mode shape of the intact structure in the second order. Neglecting the change in mode shape in Eq. (5) results in:

$$\delta\boldsymbol{\lambda}_i = \varphi_i^{\mathrm{T}}\delta\boldsymbol{K}\varphi_i \qquad (6)$$

$\delta\boldsymbol{\lambda}_i$ is known as the change in buckling load excluding the changes in mode shape.

These changes can be written as summation of changes in each element's stiffness to establish the relationship between changes in stiffness matrix and eigenvalues. If $\boldsymbol{K}_j^e$ is the share of jth element in stiffness matrix, then a new stiffness matrix can be shown as:

$$\boldsymbol{K}+\delta\boldsymbol{K} = \sum_{j=1}^{n_e}\boldsymbol{K}_j^e\left(1+\delta k_j\right) \qquad (7)$$

where n_e is the number of elements with n degree of freedom. δk_j is the change in stiffness in jth element. Therefore, δk can be described as:

$$\delta\boldsymbol{K} = \sum_{j=1}^{n_e}\boldsymbol{K}_j^e \times \delta k_j \qquad (8)$$

Equation (8) has below features:

1) The symmetry of stiffness matrix is kept.
2) The connection of nodes is kept.
3) The final changes in the global stiffness matrix are related to changes in each element.

2.2 Second Order

In the second order, the change in mode shape is considered. Therefore, Eq. (3) without deleting $\delta\varphi_i$ is shown as:

$$\delta\lambda_i = {\varphi_i}^{\mathrm{T}}\delta\boldsymbol{K}\varphi_i + {\varphi_i}^{\mathrm{T}}\delta\boldsymbol{K}\delta\varphi_i \tag{9}$$

The changes in the mode shape of the structure can be expressed as a linear combination of the mode shapes of the intact structure by using Fox's formulation.

$$\varphi_i' \cong \sum_{j=1}^{n} \alpha_{ij}\varphi_j \tag{10}$$

where α_{ij} is the linear coefficient of jth mode shape used in calculation of changes in ith mode shape.

The derivation of Eq. (1) can be described by:

$$(\boldsymbol{K}-\lambda_i\boldsymbol{K}_G)\varphi_i' = -(\boldsymbol{K}-\lambda_i\boldsymbol{K}_G)'\varphi_i \tag{11}$$

Substituting Eq. (10) in Eq. (11) results in:

$$(\boldsymbol{K}-\lambda_i\boldsymbol{K}_G)\sum_{j=1}^{n}\alpha_{ij}\varphi_j = -(\boldsymbol{K}-\lambda_i\boldsymbol{K}_G)'\varphi_i \tag{12}$$

Premultiplying Eq. (12) by $\varphi_j(k \neq i)$ followed by transporting and rearranging it yields:

$$\sum_{j=1}^{n}\alpha_{ij}{\varphi_k}^{\mathrm{T}}(\boldsymbol{K}-\lambda_i\boldsymbol{K}_G)\varphi_j = -{\varphi_k}^{\mathrm{T}}(\boldsymbol{K}-\lambda_i\boldsymbol{K}_G)'\varphi_i \tag{13}$$

The expanded form of Eq. (13) is:

$$\begin{aligned}&\alpha_{i1}{\varphi_k}^{\mathrm{T}}(\boldsymbol{K}-\lambda_i\boldsymbol{K}_G)\varphi_1 + \ldots + \alpha_{in}{\varphi_k}^{\mathrm{T}}(\boldsymbol{K}-\lambda_i\boldsymbol{K}_G)\varphi_n\\&= -{\varphi_k}^{\mathrm{T}}(\boldsymbol{K}-\lambda_i\boldsymbol{K}_G)'\varphi_i\end{aligned} \tag{14}$$

Considering the orthogonality property of mode shapes, Eq. (14) can be written as:

$$\alpha_{ik}{\varphi_k}^{\mathrm{T}}(\boldsymbol{K}-\lambda_i\boldsymbol{K}_G)\varphi_j = -{\varphi_k}^{\mathrm{T}}(\boldsymbol{K}-\lambda_i\boldsymbol{K}_G)'\varphi_i \tag{15}$$

Table 1 Column parameters

Parameter	Value
E/GPa	200
I/m^4	5.4×10^{-10}
L/m	4

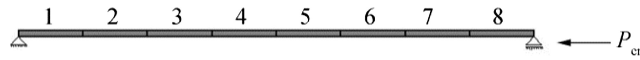

Fig. 1 Simply supported column under compressive tip load

Equation (15) can be simplified as:

$$\alpha_{ik}(\lambda_k-\lambda_i) = -{\varphi_k}^{\mathrm{T}}\left(\boldsymbol{K}'-\lambda_i\boldsymbol{K}_G'-2\lambda_i'\lambda_i\boldsymbol{K}_G\right)\varphi_i \tag{16}$$

Considering the orthogonality property of mode shapes:

$${\varphi_k}^{\mathrm{T}}\lambda_i'\lambda_i\boldsymbol{K}_G\varphi_i = 0 \tag{17}$$

Assuming $\boldsymbol{K}_G' = 0$, the rate of changes in ith mode shape can be expressed as:

$$\alpha_{ik} = \frac{-{\varphi_k}^{\mathrm{T}}\delta\boldsymbol{K}\varphi_i}{(\lambda_k-\lambda_i)} \tag{18}$$

when $i = k$, α_{ik} can be calculated by derivation of Eq. (2):

$${\varphi_i}^{\mathrm{T}}\boldsymbol{K}_G\varphi_i + {\varphi_i}^{\mathrm{T}}\boldsymbol{K}_G'\varphi_i + {\varphi_i}^{\mathrm{T}}\boldsymbol{K}_G\varphi_i' = 0 \tag{19}$$

Using the symmetry property of the stiffness matrix, Eq. (19) can be simplified as:

$$2{\varphi_i}^{\mathrm{T}}\boldsymbol{K}_G\varphi_i' = {\varphi_i}^{\mathrm{T}}\boldsymbol{K}_G'\varphi_i \tag{20}$$

Substituting φ_i' from Eq. (10) in Eq. (20) results in:

$$2{\varphi_i}^{\mathrm{T}}\boldsymbol{K}_G\sum_{j=1}^{n}\alpha_{ij}\varphi_j = \varphi_i\boldsymbol{K}_G'\varphi_i \tag{21}$$

Assuming $\boldsymbol{K}_G' = 0$ and the orthogonality property of mode shapes:

$$\alpha_{ij} = 0 \tag{22}$$

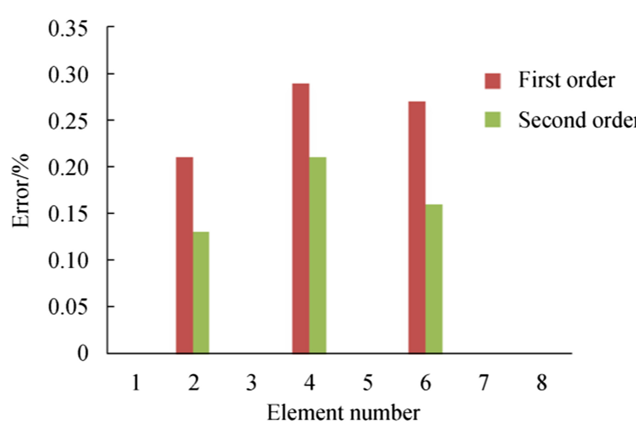

Fig. 2 First- and second-order error percentage

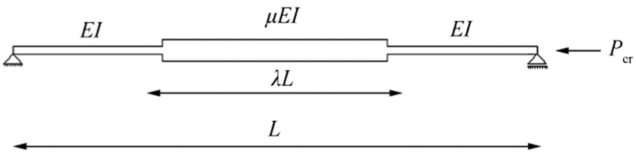

Fig. 3 Cross-section variations along the length of column

Table 3 Beam column properties

Parameter	Value
E/GPa	200
I/m^4	8.3×10^{-10}
L/m	10

Therefore, α_{ik} can be written as:

$$\begin{cases} \alpha_{ik} = \dfrac{-{\boldsymbol{\varphi}_k}^{\mathrm{T}} \delta \boldsymbol{K} \boldsymbol{\varphi}_i}{(\boldsymbol{\lambda}_k - \boldsymbol{\lambda}_i)} & i \neq k \\ \alpha_{ii} = 0 & i = k \end{cases} \quad (23)$$

Finally, the change in buckling load is given as:

$$\delta \boldsymbol{\lambda}_i = \left({\boldsymbol{\varphi}_i}^{\mathrm{T}} \delta \boldsymbol{K} \boldsymbol{\varphi}_i - {\boldsymbol{\varphi}_i}^{\mathrm{T}} \frac{{\boldsymbol{\varphi}_k}^{\mathrm{T}} \delta \boldsymbol{K} \boldsymbol{\varphi}_i}{(\boldsymbol{\lambda}_k - \boldsymbol{\lambda}_i)} \boldsymbol{\varphi}_k \right) \quad (24)$$

2.3 Stiffness and Geometric Stiffness Matrices

In analysis of the stability of structures, if the value of load causes elastic behavior, then linear elasticity theory may be used. However, if load increases largely, the structure could behave in a geometrically nonlinear manner, in a materially nonlinear manner, or their combination. For beam element stiffness and geometric stiffness, the matrices are:

$$\boldsymbol{K} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & -2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & -2L^2 & -6L & 4L^2 \end{bmatrix} \quad (25)$$

$$\boldsymbol{K}_G = \frac{T}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \quad (26)$$

Table 2 Comparison of buckling load of simply supported tapered column obtained by the proposed method and other methods

L/m	λ	μ	P_{cr}/N		
			Timoshenko	FEM	Present method
5	0.2	1.66	77.245	77.232	78.391
5	0.2	1.25	70.965	70.976	72.392
5	0.4	1.66	90.636	90.561	92.462
5	0.4	1.25	76.194	76.167	77.309
8	0.6	1.66	39.943	39.947	40.466
8	0.6	2.5	55.182	55.182	56.231
8	0.8	1.66	42.278	42.278	43.757
8	0.8	2.5	62.703	62.732	64.237

3 Numerical Results

3.1 General

Equation (24) describes the changes in buckling load due to existence of tapered member, in which the change in mode shape is considered. In similar works, vibration mode shapes are used instead of buckling mode shapes. This practice might be usable for single-span columns but not for multi-span columns, where vibration mode shapes do not have analytical calculation. Thus, employing this method is invalid for multi-span columns. Current method utilizes eigenvalue equation to calculate buckling mode shapes of uniform columns and then uses α_{ik} in Eq. (18) to calculate the buckling mode shapes of non-uniform columns.

The numerical procedure based on developed formulation was programmed on a desktop computer, and the numerical results are presented to demonstrate their versatility and accuracy in solving buckling problems of stepped columns in the following sections.

Example 1: Buckling of Simply Supported Tapered Columns

In theory, section buckling load is described in two orders: first order and second order. In the first order, the change in mode shape is neglected. In this example, error due to this neglect is investigated.

For this purpose, a simply supported column is considered. In the finite element model, the total number of elements is eight. Table 1 shows the column parameters, and the column is shown in Fig. 1.

The first- and second-order results are compared with those obtained by exact solution. The error percentage of these orders is shown in Fig. 2. The flexural rigidity of three elements is increased by 20% in differential analysis to obtain the error percentage.

As shown in Fig. 2, considering the change in mode shape increases the accuracy of calculation. Moreover, the error percentage is less regardless of the variation close to the boundaries.

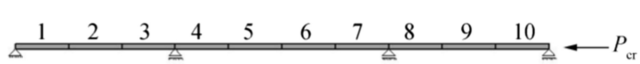

Fig. 4 Simply supported continuous column subjected to axial loading

Table 4 Comparison of buckling load of simply supported continuous column obtained by the proposed method and FEM

Cases of changes				P_{cr}/N	
Case No.	Change type	Element No.	Variation/%	FEM	Present method
1	Decrease	2 6	20 30	130.579	133.062
2	Increase	1 4	25 20	147.187	149.541
3	Decrease	3 4	15 15	142.492	144.201

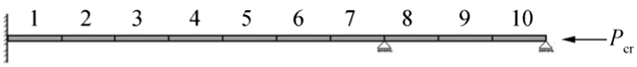

Fig. 5 Continuous column subjected to axial loading with combined boundary conditions

3.2 Example 2: Buckling of Simply Supported Tapered Column; Middle Section Fatted

Figure 3 shows a tapered column that is simply supported on the ends and subjected to axial compression P_{cr}. The unknown parameters are length of column L and flexural stiffness of column EI, where E is the Young's modulus of elasticity, $E = 200$ (GPa) and I is the column flexural moment of inertia, $I_0 = 8.3e-10\ m^4$. The middle section of this column has an abrupt change with length of λL, and the flexural stiffness changes with rate of μ. The buckling loads for different values of stiffness ratio μ and changes in length λ are presented in Table 2.

The results are compared with those obtained by finite element method and exact method presented by Timoshenko and Gere (1961). A good agreement is observed.

3.3 Example 3: Buckling of Continuous Tapered Columns with Different Boundary Conditions

In this section, two types of continuous columns are presented to investigate buckling load. Different types of boundary condition and cross section changes are used in the two columns. Table 3 shows the main parameters of both continuous columns.

In this section, the effects of boundary conditions and location of tapered member on buckling load are investigated.

Example 4: Continuous Column No. 1

For continuous column No. 1 (Fig. 4), three cases are considered. In the first case, the flexural rigidity values of elements 2 and 6 are reduced by 20% and 30%, respectively. In the second case, the flexural rigidity values of elements 1 and 4 are increased by 25% and 20%, respectively. In the third case, the stiffness of elements 3 and 17 is reduced by 15%. Table 4 shows the results obtained by the proposed method and finite element analysis performed by ANSYS.

For tapered members near the boundary conditions, the change in buckling load and error in the proposed method is low. If the variation in stiffness occurs in greater span, then more changes in buckling load are observed.

Example 5: Continuous Column No. 2

pt?>In the other continuous column (Fig. 5), boundary conditions are a combination of clamped and simply supported type. In this column, three cases are also considered. In the first case, the flexural rigidity values of elements 1 and 6 are increased by 30% and 15%, respectively. In the second case, the flexural rigidity values of elements 7 and 8 are reduced by 20% and 10%, respectively. In the third case, the stiffness values of elements 4 and 9 are increased by 30% and 20%, respectively. Table 5 shows the results obtained by the proposed method and finite element analysis performed by ANSYS.

In general, the presence of more supports decreases the span length, thereby increasing the buckling load.

Example 6: Buckling of Cantilever Tapered Column

Table 5 Comparison of buckling load of continuous column obtained by the proposed method and FEM

Cases of changes				P_{cr}/N	
Case No.	Change type	Element No.	Variation/%	FEM	Present method
1	Increase	1 6	30 15	101.681	104.528
2	Decrease	7 8	20 10	94.548	96.249
3	Increase	4 9	30 20	103.139	105.821

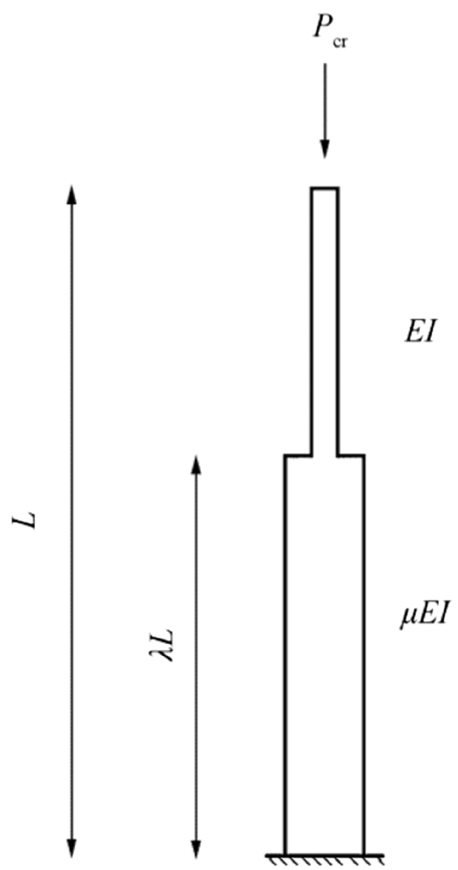

Fig. 6 Cantilever column with varying cross section

A steel tapered cantilever column used by Timoshenko and Gere (1961) as a numerical example for calculating the critical buckling load is shown in Fig. 6. The material and cross-sectional properties are as follows $E = 200$ (GPa), $I = 5.4 \times 10^{-9}$ m^4, and stiffness ratio $\mu = 1.5$. The proposed method is compared with finite element analysis performed by ANSYS and the analytic solution by Timoshenko and Gere (1961). The results of the comparison are shown in Table 6.

Table 6 Comparison of buckling load of cantilever column obtained by the proposed method and FEM

L/m	λ	P_{cr}/N		
		Timoshenko	FEM	Present method
2	0.2	762.943	762.943	764.351
2	0.3	815.218	815.166	818.332
2	0.4	866.62	866.616	870.102
2	0.5	913.425	913.424	917.754
4	0.2	190.735	190.735	192.419
4	0.3	203.792	203.791	207.421
4	0.4	216.654	216.654	220.943
4	0.5	228.355	228.356	233.149

4 Conclusion

This study proposes a new approach for calculating the buckling load of tapered columns with variable cross section by using modified mode shapes. The mode shapes of a column depend on structural stiffness parameters. The proposed method can be used for single-span and continuous columns with piecewise variation in their length. The mode shape of the damaged structure can be expressed as a linear combination of the mode shapes of the intact structure. This property is used to accurately determine buckling load. Several examples are presented to illustrate the capability of the method. The results are also compared with finite element method and Timoshenko method.

For future work, the proposed method can be used for sensitivity analysis to reduce column weight. Using sensitivity analysis, the method can detect the magnitude and location of the damaged elements to which the buckling aspect of the structure is intended. The method can also be developed for tapered columns with elastic foundation. The buckling load of tapered columns of linearly varying section must also be investigated in the future.

5 Concrete Application

Marine structures, such as ships and platforms, are composed of various elements, including beams, columns, and plates. In all of these structures, buckling is an important issue. In platforms, buckling problem is considered at the installation phase and in the piles due to the axial force caused by the impact. In cases where the platform's buckling capacity should be increased, the proposed method can be used to find the element or elements to increase the stiffness.

The ship's plates are referred to as other examples. The presence of variable cross sections in the ship plates that are strongly corroded also significantly increases the buckling capacity and flexural stiffness.

Nomenclature

$\mathbf{K}$	stiffness matrix
$\mathbf{K}_G$	geometric stiffness matrix
$\boldsymbol{\lambda}_i$	eigenvalue of the structure
φ_i	mode shape of the structure
δ_{ij}	Kronecker's delta
$\delta\boldsymbol{\lambda}_i$	variation of eigenvalue
$\delta\boldsymbol{\varphi}_i$	variation of mode shape
$\delta\boldsymbol{K}_i$	variation of stiffness matrix
α_{ij}	linear coefficient of jth mode shape
φ_i'	changes in mode shape of the structure
E	Young's modulus of elasticity
I	column flexural moment of inertia
T	internal load of elements
L	beam length
P_{cr}	critical buckling load of column

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