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Scattering of Oblique Surface Water Waves by Thin Vertical Barrier Over Undulating Bed Topography

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Abstract: The present study deals with the scattering of oblique surface water waves by small undulation on the bottom in the presence of a thin vertical barrier. Here, three different configurations of vertical barriers are investigated. Perturbation analysis is employed to determine the physical quantities, namely, the reflection and transmission coefficients. In this analysis, many different Boundary Value Problems (BVPs) are obtained out of which the first two bvps are considered. The zeroth order bvp is solved with the aid of eigenfunction expansion method. The first order reflection and transmission coefficients are derived in terms of the integrals by the method of the Green's integral theorem. The variation of these coefficients is plotted and analyzed for different physical parameters. Furthermore, the energy balance relation, an important relation in the study of water wave scattering, is derived and checked for assuring the correctness of the numerical results for the present problem.

Keywords: oblique wave scattering, bottom undulation, vertical barrier, eigenfunction expansion, Green's integral theorem, reflection and transmission coefficients

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1 Introduction

The problems of scattering of water waves by vertical barriers have been drawing a great attention of many researchers for a long time. These problems are important because of their engineering applications such as wavemakers and breakwaters which protect a harbor from the rough sea. Dean (1945) used the complex variable technique to obtain the reflection coefficients from the linearized solution of water wave scattering in the presence of a thin vertical barrier. Ursell (1947) obtained the solution of the problem of water wave diffraction by thin vertical barrier partially immersed in deep water by making use of the singular integral equation along with Havelock's expansion. Porter (1972) derived the solution of the problem involving wave transmission through a gap in a vertical barrier in deep water by using the complex variable technique as well as Green's integral theorem. Banerjea *et al.* (1996) utilized the one-term and multi-term Galerkin approximation to evaluate the reflection coefficient for the problem considered by Porter (1972). Losada *et al.* (1992) obtained the reflection and transmission coefficients of the problem involving scattering of water waves by four different types of thin vertical barriers using eigenfunction expansion method. Mandal and Dolai (1994), and Porter and Evans (1995) derived the solution of the problem of Losada *et al.* (1992) by using the Galerkin approximation. Mandal and Chakrabarti (1999) employed the Galerkin approximation to determine the approximate solutions of a number of water wave scattering problems involving thin vertical barriers in deep water as well as finite depth of water. Sahoo *et al.* (2000), and Lee and Chawang (2000) analyzed the problem involving the scattering of water waves by permeable vertical barrier using eigenfunction expansion method.

The above works were focused on the scattering of surface water waves by barrier only. On the other hand, the problems involving the scattering of surface water waves by a geometrical disturbance or an obstacle at the bottom are interesting for their possible application in the areas of coastal and marine engineering. The problem consisting of the reflection of surface waves by bottom undulation has been drawing a great attention as its mechanism is important in the development of shore-parallel bars. Such problem was solved by Miles (1981) utilizing the perturbation theory and finite cosine transformation while Davies (1982) employed perturbation theory and Fourier transform technique to obtain the reflection coefficient. Davies and Heathershaw (1984) compared the theoretical results of Davies (1982) by conducting the experiment in a wave tank. Mandal and Basu (1990) generalized the problem of Miles (1981) with inclusion of surface tension at the free surface while Kirby (1993) examined the problem where the incident wave is not necessarily close to the resonant frequency. Martha and Bora (2007) studied the problem for a number of practical examples of bottom undulations. Mandal and Gayen (2006) studied the problem of diffraction of surface water waves by vertical barrier and bottom undulation.

It may be underlined that the combination of vertical barrier and the bottom undulation will serve as an effective breakwater. So it is an endeavor to consider scattering problems involving both barrier and bottom undulation, which help in the real and practical situations. Also, oblique

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incidence of surface water waves is more natural than the normal incidence. In addition to these, this paper highlights about the applicability of mathematical techniques to solve the mixed Boundary Value Problem (BVP) arising from the scattering problem studied here.

In the present paper, a mixed BVP occurs in a natural way while examining the scattering of oblique water waves by a thin vertical barrier over an undulating bottom topography. With the help of perturbation expansion, many bvps are obtained out of which the first two bvps are considered. The BVP-I (zeroth order) corresponds to the problem of scattering of surface waves by barrier over flat bed. This bvp is solved utilizing the eigenfunction expansion method followed by least-squares and QR-factorization giving rise to zeroth order reflection and transmission coefficients. The BVP-II represents the radiation problem involving the first order reflection and transmission coefficients. Using the Green's integral theorem, these coefficients are obtained in terms of the integrals involving bottom undulation and the solution of zeroth order BVP. The effect of different parameters involved in the present study, is examined through different graphs. The energy balance relation for the given problem is investigated and checked for assuring the integrity of the theoretical and numerical results of the reflection and transmission coefficients.

2 Formulation of the problem

A right handed Cartesian co-ordinate system is taken in which the *xz*-plane is the undisturbed free surface of the fluid and the *y*-axis is directed positive vertically downward. The bottom with a small deformation is described by $y = h + \varepsilon c(x)$ where $c(x)$ is continuous bounded function interpreting the form of the bottom undulation and $c(x) \to 0$ as $|x| \to \infty$. Here, h denotes the uniform finite depth of the fluid far to either side of the undulation of the bottom and the non-dimensional number $\varepsilon(\ll 1)$ gives the quantum of smallness of the undulation. Let a thin vertical barrier B (say) is fixed in the fluid whose position may be described as follows:

- Case-I: $x = 0,\ 0 \le y \le a$ (a partially immersed barrier)
- Case-II: $x = 0,\ b \le y \le h + \varepsilon c(0)$ (a bottom standing barrier)
- Case-III: $x = 0,\ 0 \le y \le d,\ d + e \le y \le h + \varepsilon c(0)$ (a barrier with a gap)

Partial differential equation and boundary conditions:

It is assumed that the fluid is incompressible, inviscid and the motion to be irrotational and simple harmonic in time. Consider a progressive wave train represented by the velocity potential $\mathrm{Re}\{\hat{\phi}(x,y)\mathrm{e}^{\mathrm{i}(\nu z - \sigma t)}\}$ is incident obliquely upon the bottom undulation and the vertical barrier, where

$$\hat{\phi}(x,y) = \psi_0(y)\mathrm{e}^{\mathrm{i}\mu x} \tag{1}$$

with $\psi_0(y) = N_0^{-1}\cosh \hat{k}_0(h-y)$ and $N_0 = [(1 + (\sinh^2 \hat{k}_0 h)/Kh)/2]^{1/2}$, $k = \hat{k}_0$, the wave number of the incident wave, is the positive real root of the transcendental equation $K - k\tanh kh = 0$, $\mu = \hat{k}_0 \cos\theta$, $\nu = \hat{k}_0 \sin\theta$, θ being the angle of incidence of wave train with mean free surface of the fluid, $K = \sigma^2 / g$, σ is the angular frequency of the incoming water wave train with time dependence $\mathrm{e}^{-\mathrm{i}\sigma t}$ and g is the acceleration due to gravity.

Due to uniformity in the *z*-direction and the periodicity in time, the velocity potential which describes the fluid motion can be expressed as $\mathrm{Re}\{\phi(x,y)\mathrm{e}^{\mathrm{i}(\nu z - \sigma t)}\}$. Then, assuming the linear theory, the complex valued potential $\phi(x,y)$ satisfies the Helmholtz's Eq. (2) and the boundary conditions (3)–(8):

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \nu^2 \phi = 0, \quad \text{in the fluid region} \tag{2}$$

Free surface condition:

$$\frac{\partial \phi}{\partial y} + K\phi = 0, \qquad \text{on } y = 0 \tag{3}$$

Bottom condition:

$$\frac{\partial \phi}{\partial n} = 0, \qquad \text{on } y = h + \varepsilon c(x) \tag{4}$$

Condition on barrier:

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=0^-} = \left.\frac{\partial \phi}{\partial x}\right|_{x=0^+} = 0, \qquad \text{on } x = 0, y \in B \tag{5}$$

Conditions across gap:

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=0^-} = \left.\frac{\partial \phi}{\partial x}\right|_{x=0^+}, \qquad \text{on } x = 0,\ y \in G \tag{6}$$

$$\phi|_{x=0^-} = \phi|_{x=0^+}, \qquad \text{on } x = 0,\ y \in G \tag{7}$$

Far field condition:

$$\phi(x,y) \to \begin{cases} \hat{\phi}(x,y) + R\hat{\phi}(-x,y) & \text{as} \quad x \to -\infty \\ T\hat{\phi}(x,y) & \text{as} \quad x \to \infty \end{cases} \tag{8}$$

where $\partial / \partial n$ is the outward normal derivative on the bottom undulation, G denotes the gap, so that $B \cup G$ is $[0, h + \varepsilon c(0)]$ and this gap depends on the configurations of the barrier; R and T represent the reflection and transmission coefficients respectively which are to be determined here along with $\phi(x,y)$.

3 Method of solution

The bottom condition $\partial\phi / \partial n = 0$ on $y = h + \varepsilon c(x)$ can be approximated up to the first order of the small parameter ε as

$$\frac{\partial \phi}{\partial y} - \varepsilon\left\{\frac{\mathrm{d}}{\mathrm{d}x}[c(x)\phi_x] + c(x)\phi_{zz}\right\} + O(\varepsilon^2) = 0, \text{ on } y = h \tag{9}$$

The approximate boundary condition (9) suggests that $\phi(x,y)$, R and T can be expressed in terms of ε as given by

$$\left.\begin{aligned} \phi(x,y) &= \phi_0 + \varepsilon\phi_1 + O(\varepsilon^2) \\ R &= R_0 + \varepsilon R_1 + O(\varepsilon^2) \\ T &= T_0 + \varepsilon T_1 + O(\varepsilon^2) \end{aligned}\right\} \tag{10}$$

It must be noted that such a perturbation expansion ceases to be valid at Bragg resonance when the reflection coefficient becomes much larger than the dimensionless undulation parameter ε, as noticed by Mei (1985). Also, this theory is valid only for the infinitesimal reflection and away from resonance. It may be reminisced that the Bragg resonance occurs when the wave number of the bottom undulation is twice the wave number of free surface. However, Mei (1985) overcomes this Bragg resonance situation by formulating wave evolution and the reflection theory at and near the Bragg resonance condition for shore-parallel bars. Since, the bottom undulations are small, there is no concern of large reflection in our work. Hence, the perturbation expansion as given by the relation (10) is valid throughout the present work.

On substituting the expressions of $\phi(x,y)$, R and T from relation (10) into the relations (2), (3) and (5)–(9), then equating the coefficients of ε^0 and ε from both sides of all equations, we have the following BVPs:

BVP-I: The function $\phi_0(x,y)$ satisfies

$$\frac{\partial^2\phi_0}{\partial x^2} + \frac{\partial^2\phi_0}{\partial y^2} - \nu^2\phi_0 = 0, \quad \text{in the fluid region} \tag{11}$$

$$\frac{\partial\phi_0}{\partial y} + K\phi_0 = 0, \quad \text{on} \quad y = 0 \tag{12}$$

$$\frac{\partial\phi_0}{\partial y} = 0, \quad \text{on } y = h \tag{13}$$

$$\left.\frac{\partial\phi_0}{\partial x}\right|_{x=0^-} = \left.\frac{\partial\phi_0}{\partial x}\right|_{x=0^+} = 0, \quad \text{on } x = 0, y \in B \tag{14}$$

$$\left.\frac{\partial\phi_0}{\partial x}\right|_{x=0^-} = \left.\frac{\partial\phi_0}{\partial x}\right|_{x=0^+}, \quad \text{on } x = 0,\ y \in G \tag{15}$$

$$\phi_0|_{x=0^-} = \phi_0|_{x=0^+}, \quad \text{on } x = 0, \quad y \in G \tag{16}$$

$$\phi_0(x,y) \to \begin{cases} (\mathrm{e}^{\mathrm{i}\mu x} + R_0\mathrm{e}^{-\mathrm{i}\mu x})\psi_0(y) & \text{as} \quad x \to -\infty \\ T_0\mathrm{e}^{\mathrm{i}\mu x}\psi_0(y) & \text{as} \quad x \to \infty \end{cases} \tag{17}$$

BVP-II: The function $\phi_1(x,y)$ satisfies

$$\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} - \nu^2\phi_1 = 0, \quad \text{in the fluid region} \tag{18}$$

$$\frac{\partial\phi_1}{\partial y} + K\phi_1 = 0, \quad \text{on} \quad y = 0 \tag{19}$$

$$\frac{\partial\phi_1}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left[c(x)\frac{\partial\phi_0}{\partial y}\right] + c(x)\phi_{zz}, \quad \text{on } y = h \tag{20}$$

$$\left.\frac{\partial\phi_1}{\partial x}\right|_{x=0^-} = \left.\frac{\partial\phi_1}{\partial x}\right|_{x=0^+} = 0, \quad \text{on } x = 0, y \in B \tag{21}$$

$$\phi_1|_{x=0^-} = \phi_1|_{x=0^+}, \quad \text{on } x = 0, y \in G \tag{22}$$

$$\left.\frac{\partial\phi_1}{\partial x}\right|_{x=0^-} = \left.\frac{\partial\phi_1}{\partial x}\right|_{x=0^+}, \quad \text{on } x = 0,\ y \in G \tag{23}$$

$$\phi_1(x,y) \to \begin{cases} R_1\mathrm{e}^{-\mathrm{i}\mu x}\psi_0(y) & \text{as} \quad x \to -\infty \\ T_1\mathrm{e}^{\mathrm{i}\mu x}\psi_0(y) & \text{as} \quad x \to \infty \end{cases} \tag{24}$$

Here, the BVP-I corresponds to the problem of scattering of water waves by thin vertical barrier in water of finite depth h. The solution for $\phi_0(x,y)$ can be expressed as

$$\phi_0(x,y) \to \begin{cases} (\mathrm{e}^{\mathrm{i}\mu x} + R_0\mathrm{e}^{-\mathrm{i}\mu x})\psi_0(y) + \sum_{n=1}^{\infty} A_n\mathrm{e}^{s_n x}\psi_n(y) & \text{as} \quad x < 0, \\ T_0\mathrm{e}^{\mathrm{i}\mu x}\psi_0(y) + \sum_{n=1}^{\infty} B_n\mathrm{e}^{-s_n x}\psi_n(y) & \text{as} \quad x > 0, \end{cases} \tag{25}$$

where $s_n = \sqrt{k_n^2 + \nu^2}$, $\psi_n(y) = N_n^{-1}\cos k_n(h-y)$ with $N_n = [(1 - (\sin^2\hat{k}_0 h)/Kh)/2]^{1/2}$, and $k = k_n$, $(n = 1, 2, \ldots)$ are the positive real roots of the transcendental equation $K + k\tan kh = 0$. Here R_0, T_0, A_n and B_n are unknown complex constants to be determined to obtain the zeroth order velocity potential $\phi_0(x,y)$ completely. It is noted that the set of eigenfunctions $\psi_n(y)$ $(n = 0, 1, 2, \ldots)$ form a complete orthonormal set with

$$\frac{1}{h}\int_0^h \psi_n(y)\psi_m(y)\mathrm{d}y = \delta_{nm}$$

where δ_{nm} is the Kronecker delta.

Now, the relation (15) holds good for $0 < y < h$, because the horizontal velocity vanishes on the barrier. Using Havelock's inversion formula (Ursell, 1947) in the relation $\partial\phi_0/\partial n|_{x=0^-} = \partial\phi_0/\partial n|_{x=0^+}$, $0 < y < h$, we obtain

$$R_0 + T_0 = 1 \text{ and } A_n = -B_n \tag{26}$$

Further, using the boundary condition (16) that the pressure has to be continuous and making use of the relation (26), we have

$$\frac{1}{h}\psi_0(y) + \frac{1}{h}\sum_{n=0}^{\infty} A_n\psi_n(y), \quad \text{on } x = 0,\ y \in G \tag{27}$$

where

$$A_0 = R_0 - 1, k_0 = -\mathrm{i}\hat{k}_0 \tag{28}$$

Now, using the boundary condition (14), we get

$$\sum_{n=0}^{\infty} A_n(s_n h)\frac{1}{h}\psi_n(y) = 0, \quad \text{on } x = 0, y \in B \tag{29}$$

where $s_0 = -\mathrm{i}\mu$.

The relations (27) and (29) are known as dual series relations and can be combined to make one mixed boundary condition as given by

$$F(y) = 0, \quad 0 < y < h \tag{30}$$

where

$$F(y) \to \begin{cases} \dfrac{1}{h}\psi_0(y) + \dfrac{1}{h}\sum_{n=0}^{\infty} A_n\psi_n(y), & \text{on } y \in G \\ \sum_{n=0}^{\infty} A_n(s_n h)\dfrac{1}{h}\psi_n(y), & \text{on } y \in B \end{cases}$$

The relation (30) represents an over-determined system of equations involving countably infinite number of unknowns A_n $(n = 0, 1, 2, \ldots)$ and uncountably infinite number of equations for values of y, belonging to G or B. Such an over-determined system can be solved by using an appropriate method. In the present paper, we have used method of analytic least-squares as described below. It may be noted that on increasing the number of equations in the over-determined system (30), the coefficient matrix associated with the over-determined system of equations may be rank deficient. To avoid such situation, the QR-factorization is used to solve the system of normal equations to obtain the unknowns A_n.

Now, the least-squares error is established as

$$\text{Error}=\left(\int_0^h |F(y)|^2\,\mathrm{d}y\right)^{1/2} = \left(\int_{y\in G} |F(y)|^2\,\mathrm{d}y + \int_{y\in B} |F(y)|^2\,\mathrm{d}y\right)^{1/2} \quad (31)$$

Minimizing the error with respect to A_n establishing the following normal equations:

$$\int_{y\in G} F^*(y)\frac{\partial F(y)}{\partial A_m}\mathrm{d}y + \int_{y\in B} F^*(y)\frac{\partial F(y)}{\partial A_m}\mathrm{d}y = 0,\ m = 0, 1, 2, \ldots, \quad (32)$$

where $F^*(y)$ is the complex conjugate of $F(y)$.

On substituting the value of F^* and the derivatives of F with respect to the unknowns, the relation (32) gives a system of complex matrix equations which is given by

$$\sum_{n=0}^{\infty} A_n^*\left[\frac{1}{h}\int_{y\in G}\psi_n(y)\psi_m(y)\mathrm{d}y + \frac{(s_m h)(s_n^* h)}{h}\int_{y\in B}\psi_n(y)\psi_m(y)\mathrm{d}y\right] = -\frac{1}{h}\int_{y\in G}\psi_0(y)\psi_m(y)\mathrm{d}y,\ m = 0, 1, 2, \ldots, \quad (33)$$

For each case (given in Section 2), truncating the series for n and m to a finite number of terms N, a system of $N+1$ simultaneous equations with $N+1$ unknowns A_n $(n = 0, 1, 2, \ldots, N)$ is obtained and is solved here with the help of the QR-factorization to produce the numerical values of A_n $(n = 0, 1, 2, \ldots, N)$.

Now, the BVP-II represents the radiation problem containing $\phi_0(x, y)$ the solution of BVP-I. Applying Green's integral theorem to the functions $\phi_0(x,y)$ and $\phi_1(x,y)$ on the region bounded by

$$y = 0, 0 < x \le X;\ x = 0^+, y \in B;\ x = 0^-, y \in B;\ y = 0, -X \le x < 0;$$
$$x = -X, 0 \le y \le h;\ y = h, -X \le x \le X;\ x = X, 0 \le y \le h$$

where X is positive, large and tends to infinity, we obtain

$$R_1 = \frac{1}{2\mathrm{i}\mu}\int_{-\infty}^{\infty} c(x)\left(\frac{\partial\phi_0(x,h)}{\partial x}\right)^2\mathrm{d}x \quad (34)$$

Similarly, applying Green's integral theorem to the functions $\phi_0(-x, y)$ and $\phi_1(x, y)$ in the same region, we have

$$T_1 = -\frac{1}{2\mathrm{i}\mu}\int_{-\infty}^{\infty} c(x)\left(\frac{\partial\phi_0(x,h)}{\partial x}\right)\left(\frac{\partial\phi_0(-x,h)}{\partial x}\right)\mathrm{d}x \quad (35)$$

4 Validation of the results

It should be realized that in the absence of vertical barrier, the present problem reduces to the problem of scattering of water waves by bottom undulation only. Further, it is noted that in the absence of the barrier, the solution $\phi_0(x, y)$ of BVP-I represents the progressive wave (incident wave here) $\hat{\phi}(x, y)$. So, to validate the present results, we derive the results by assuming $\phi_0(x, y) = \hat{\phi}(x, y)$ and θ=0 (corresponds to normal incidence). With these assumptions, we found from relations (34) and (35) that

$$R_1 = \frac{-2\mathrm{i}\hat{k}_0^2}{2\hat{k}_0 h + \sinh 2\hat{k}_0 h}\int_{-\infty}^{\infty} c(x)\mathrm{e}^{2\mathrm{i}\hat{k}_0 x}\mathrm{d}x \quad (36)$$

$$T_1 = \frac{2\mathrm{i}\hat{k}_0^2}{2\hat{k}_0 h + \sinh 2\hat{k}_0 h}\int_{-\infty}^{\infty} c(x)\mathrm{d}x \quad (37)$$

which exactly match with the results of available in literature (Davies and Heathershaw, 1984; Mandal and Basu, 1990, for normal incidence and negligible surface tension).

The R_1 and T_1 given in relations (34) and (35) further evaluated for a particular form of the shape function.

5 Particular form of the bottom undulation

In this Section, we consider a particular form of the shape function $c(x)$ to determine the reflection and transmission coefficients respectively, given in the relations (34) and (35). The shape function $c(x)$ is considered in the form of a patch of sinusoidal ripples because the functional form of the uneven bottom closely corresponds to some obstacles which occur in a natural way and formed at the bottom due to the alluviation and ripple growth of sands.

The patch of sinusoidal ripples can be expressed as

$$c(x) = \begin{cases} c_0 \sin\lambda x, & -l \le x \le l, \\ 0, & \text{otherwise}, \end{cases} \quad (38)$$

where $l = M\pi/\lambda$, c_0 is amplitude and λ is the wave number of the sinusoidal ripples and M is a positive integer. The patch consists of M ripples, each having the same wave number λ. Substituting relation (38) and (25) (for $x > 0$) into the relation (35), T_1 vanishes identically. Further, substituting the relation (38) and (25) (for $x < 0$) into the relation (34), R_1 becomes

$$R_1 = \frac{c_0\mu(R_0-1)}{2N_0^2}\left\{\frac{\sin(\lambda-2\mu)l}{\lambda-2\mu} - \frac{\sin(\lambda+2\mu)l}{\lambda+2\mu}\right\} + \frac{\mathrm{i}c_0\mu R_0}{2N_0^2}\times$$
$$\left\{\frac{2(1-\cos\lambda l)}{\lambda} - \frac{2\lambda}{\lambda^2-4\mu^2} + \frac{\cos(\lambda-2\mu)l}{\lambda-2\mu} + \frac{\cos(\lambda+2\mu)l}{\lambda+2\mu}\right\} +$$
$$\frac{\mathrm{i}c_0}{N_0}\sum_{n=1}^{\infty}\left[\frac{s_n}{(\lambda-\mu)^2+s_n^2} - \frac{s_n}{(\lambda+\mu)^2+s_n^2} + \right. \tag{39}$$
$$\left\{\frac{(\lambda-\mu)\sin(\lambda-\mu)l - s_n\cos(\lambda-\mu)l}{(\lambda-\mu)^2+s_n^2} - \right.$$
$$\left.\left.\frac{(\lambda+\mu)\sin(\lambda+\mu)l - s_n\cos(\lambda+\mu)l}{(\lambda+\mu)^2+s_n^2}\right\}\mathrm{e}^{-s_n l}\right]\frac{s_n A_n}{N_n}$$

From relation (39), it is observed that when $\lambda = 2\mu$, the highest peak value of the first order reflection coefficient R_1 becomes a constant multiple of M the number of ripples in the patch and it varies linearly with M To visualize the effects of the various parameters on the reflection coefficient, numerical computation of the reflection coefficient R_1 given by (39) is discussed in section 6.

6 Results and discussion

A MATLAB program is written and is used to investigate the effects of the various parameters such as barrier length, number of ripples, the amplitude of the ripple, the angle of incidence and the gap below/above the barrier on the reflection coefficient $|R_1|$ given by relation (39) for a patch of sinusoidal ripples on the sea-bed. The main aim of the present investigation is to observe how the incident wave energy is transformed, after scattered by both obstacles, into the reflected and transmitted waves. The unknowns A_0 (hence R_0 from (28)), $A_1, A_2, \ldots A_N$ are computed from (33) and then $|R_1|$ is computed from (39) and plotted in the Figs. (2)–(4) for different values of different dimensionless parameters. The relation of $|R_1|$ is non-dimensionalized by using h as the length scale.

In the numerical computation, the value of $Kh = 1$, angle of incidence $\theta = \pi/4$, the wave number of the ripple $\lambda h = 1$, amplitude of the ripple $c_0/h = 0.1$ and number of ripples M=1 are kept fixed unless otherwise mentioned in the text.

A set of numerical values of $|R_1|$ is presented in Table 1 for $N = 5, 8, 10, 15$ and 20 with different values of Kh and for fixed values of $a/h, c_0/h$ and M. For numerical computation, the number of evanescent modes $N = 5$ is kept fixed since the results for $|R_1|$ are correct up to two decimal places for $N \geq 5$ as shown in Table 1.

6.1 Validation

To validate the present model, the present results are compared with the results available in the literature. It should be noted that when the length of the vertical barrier $a/h = 0$, the present problem reduces to the problem of scattering of water waves by the bottom undulation only (Davies and Heathershaw, 1984). Taking $c_0/h = 0.1$, $M = 2, \theta = 0$ and $a/h = 0$, the present result for reflection coefficient $|R_1|$ from (39) is depicted in Fig. 1 and compared with the known result (Davies and Heathershaw, 1984). The figure shows that both results are exactly the same.

Table 1 Values of $|R_1|$ against Kh for a/h=0.2, c_0/h=0.1 and $M=1$

Kh	N=5	N=8	N=10	N=15	N=20
0.05	0.017 8	0.017 4	0.017 2	0.017 0	0.016 9
0.10	0.032 0	0.031 4	0.031 1	0.030 8	0.030 6
0.30	0.069 4	0.068 3	0.067 7	0.067 0	0.066 7
0.50	0.082 1	0.080 3	0.079 4	0.078 3	0.077 8

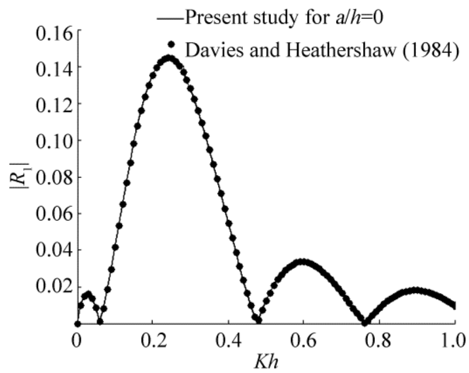

Fig. 1 $|R_1|$ against Kh for fixed M=2, a/h=0, c_0/h=0.1, θ=0 and λh=1

6.2 Effect of system parameters on reflection coefficient for different configurations of the barrier

Case I: A partially immersed barrier

In Fig. 2, the absolute value of the first order reflection coefficient $|R_1|$ is plotted against Kh for the different number of ripples, $M = 2, 4, 7$ with $\theta = 0$ (normal incidence) and $\pi/4$. In this case, for all curves, the barrier length is fixed at $a/h = 0.2$. The curve which corresponds to $M = 2$, the global maximum value (the largest value of $|R_1|$ over all values of Kh) of $|R_1|$ is 0.1334 attained at $\mu h = 0.5046$ (when $Kh = 0.44$) and for the curve corresponding to $M = 4$, the global maximum value of $|R_1|$ is 0.277 8 attained at $\mu h = 0.5046$ (when $Kh = 0.44$). Similarly for the curve corresponding to $M = 7$, the global maximum value of $|R_1|$ is 0.4688 attained at $\mu h = 0.4996$ (when $Kh = 0.43$). It is clear from this figure that the global maximum value of the reflection coefficient $|R_1|$ is attained when λh becomes approximately twice μh and the global maximum value of $|R_1|$ increases as M (number of ripples) increases. This result is plausible from the physical point of view that more number of ripples present at the bottom will produce more reflection. It is also observed that the reflection coefficient becomes more oscillatory and the number of zeros is increasing with increasing the number of ripples. These developments are due to the multiple interaction of the incident wave between the free surface, the barrier and the sinusoidal ripples. To visualize the difference between the results of oblique and normal incidences, the graph for $|R_1|$ is also plotted in Fig. 2 for $\theta = 0$ (normal incidence) with $M = 2$. It can be noticed that the global maximum value of the reflection coefficient for the normal incident waves is higher in comparison to that of oblique incident waves. This agrees with physical intuition that more energy will transfer from the incident wave to reflected wave if the wave is incident normally to the obstacles.

Fig. 3 shows the variation of the absolute value of the first order reflection coefficient $|R_1|$ against the barrier length a/h for the different values of ripple amplitude c_0/h. It can be seen that the value of $|R_1|$ is increasing as the barrier length is increasing. This agrees with the physical understanding, longer the barrier will certainly increase the reflection. Also, from this figure, it can be seen that the reflection coefficient is increasing with increasing ripple amplitude c_0/h, which agrees with the physical intuition that high ripple amplitude will produce more reflection.

Case II: A bottom standing barrier

In Fig. 4, the reflection coefficientcis $|R_1|$ depicted against Kh for different number of ripples $M = 2, 4, 7$ with gap $b/h = 0.2$. From this figure, it is clear that the global maximum value of the reflection coefficient is increasing with increasing the number of ripples M. This agrees with the physical intuition of the problem as discussed in the previous case.

In Fig. 5, the reflection coefficient $|R_1|$ is plotted against the gap b/h for different values of the ripple amplitude $c_0/h = 0.10, 0.15, 0.25$. In this figure, it can be remarked that the reflection coefficient drops with respect to the gap b/h but its value is increasing with increasing the value of ripple amplitude. It can also be noticed that the reflection coefficient drops slowly for the smaller value of the ripple amplitude. The case II with the bottom standing barrier is a complementary case of case I, i.e., in place of the barrier in case I, there is a gap in case II and in place of gap in case I, there is barrier in case II. Thus, the results from Fig. 5 of case II would be complementary of the results from the Fig. 3 of case I. These complementary observations are found exactly from Fig. 5 and are plausible from the physical understanding that the more gap causes less reflection and high ripple amplitude effects more reflection.

In Fig. 6, the absolute value of the first order reflection coefficient $|R_1|$ is plotted against the angle of incidence θ for the gap $b/h = 0.2$. The continuous decrease in the value of the reflection coefficient with respect to the angle of incidence is observed from this figure, which is plausible from the physical point of view that less energy will transfer from the incident wave to reflected wave if the angle of incidence increases.

Case III: A barrier with gap

In Fig. 7, the first order reflection coefficient $|R_1|$ is plotted against Kh for different number of ripples $M = 2, 4, 7$ with $d/h = 0.2$ and $e/h = 0.3$. The figure establishes the same characteristics with the number of ripples as observed in the previous cases.

Fig. 8 shows the effect of the amplitude of the bottom undulation c_0/h on the absolute value of first order reflection coefficient $|R_1|$ for different values of the gap $e/h = 0.20, 0.35, 0.50$. From Fig. 8, it is observed that the reflection coefficient is increasing while increasing the value of ripple amplitude. But the value of the reflection coefficient is decreasing while increasing the gap and this result agrees with the physical intuition.

In Fig. 9, the first order reflection coefficient is depicted against the angle of incidence for $d/h = 0.2, e/h = 0.2$. It shows the similar behavior of the reflection coefficient with the angle of incidence as discovered in the previous cases.

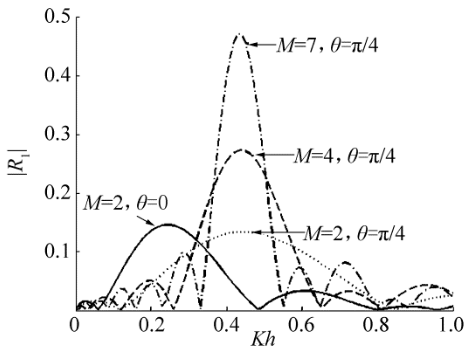

Fig. 2 $|R_1|$ **versus** Kh **for different values of** M**=2, 4, 7 with fixed** a/h**=0.2,** c_0/h**=0.1 and** λh **=1**

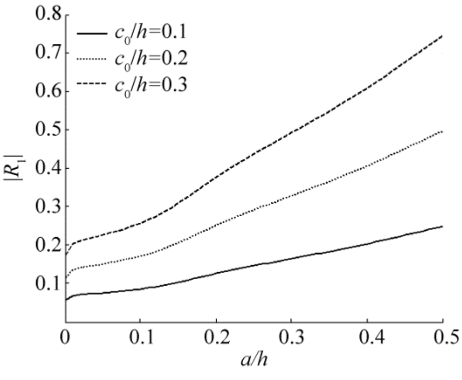

Fig. 3 $|R_1|$ **versus** a/h **for different values of** c_0/h**=0.1, 0.2, 0.3 with** Kh**=1,** M**=1,** λh **=1 and** θ**=π/4**

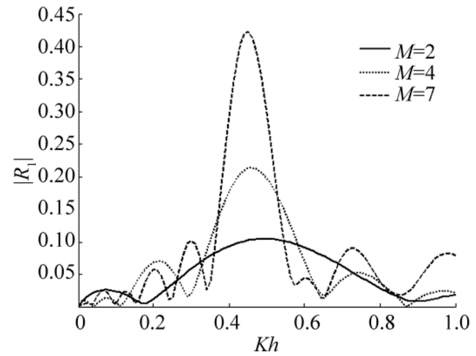

Fig. 4 **$|R_1|$ versus *Kh* for different number of ripples *M*=2, 4, 7 with *b/h*=0.2, c_0/h=0.1, λh =1 and θ=π/4**

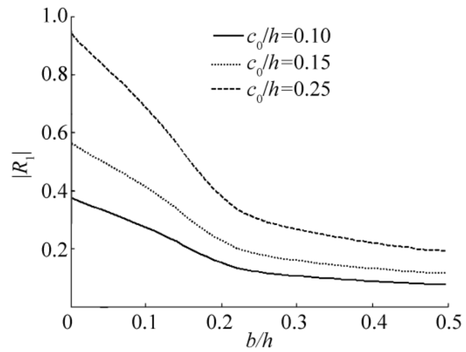

Fig. 5 **$|R_1|$ versus gap *b/h* for different values of c_0/h=0.10, 0.15, 0.25 with *Kh*=1, *M*=1, λh =1and θ=π/4**

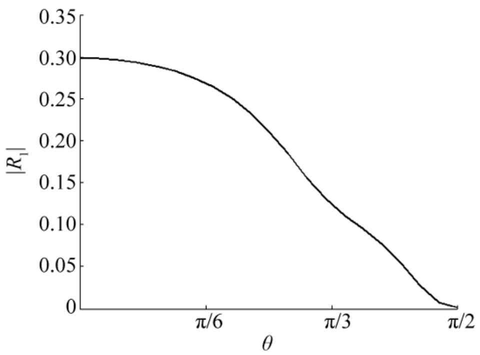

Fig. 6 **$|R_1|$ versus θ for *b/h*=0.2, c_0/h=0.1, *Kh*=1, λh=1 and *M*=1**

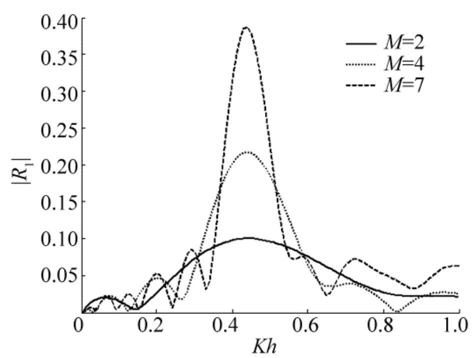

Fig. 7 **$|R_1|$ versus *Kh* for different number of ripples *M*=2, 4, 7 with *d/h*=0.2, c_0/h=0.1, *e/h*=0.3, λh =1 and θ=π/4**

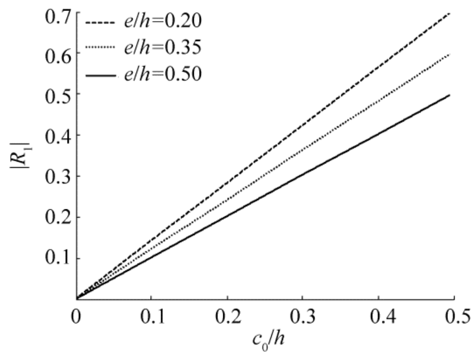

Fig. 8 **$|R_1|$ versus c_0/h for different values of *e/h*=0.20, 0.35, 0.50 with *Kh*=1, *M*=1, *d/h*=0.1, λh =1 and θ=π/4**

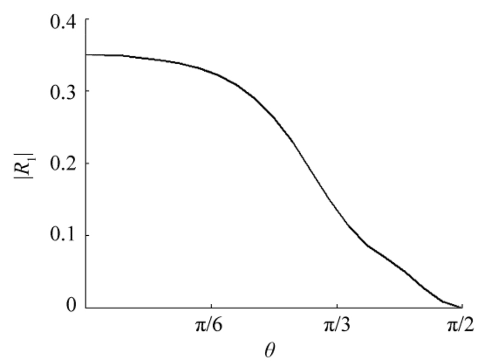

Fig. 9 **$|R_1|$ versus θ for *d/h*=0.2, *e/h*=0.2, c_0/h=0.1, *Kh*=1, λh=1 and *M*=1**

7 Energy balance relation

It is found from the literature (Chakrabarti and Martha, 2009; Chakrabarti and Mohapatra, 2013 etc.) that among many important results in the theoretical study of water waves, the special relation, namely, the "energy identity relation" plays an important role in the water wave scattering problems for checking the correctness of the results. The energy identity, in the cases when one has to rely on the numerical results for these coefficients, confirms the correctness of the numerical results. Using Green's integral theorem involving complex velocity potential and its complex conjugate, the energy identity relation for the present problem can be derived as

$$|R|^2+|T|^2=1 \tag{40}$$

Using relation (10), the numerical values of $|R|,|T|$ and hence, $|R|^2+|T|^2$ are evaluated for different values of Kh and presented in Tables 2–4 for the three different types of barrier respectively, to verify the energy balance relation (40). In all cases, the parameters $c_0/h=0.1$, $M=1$ and θ=π/4 are kept fixed.

The last column of the Tables 2–5 shows that each value of $|R|^2+|T|^2$ for each N is nearly equal to 1, that is, the reflection and transmission coefficients obtained by the method employed here satisfy the energy balance relation (40) almost accurately.

It is observed from the table 2 (for $\varepsilon = 0.1$) and Table 5 (for $\varepsilon = 0.001$) that the results obtained for $|R|$ and $|T|$ are correct up to two decimal places, satisfying the energy balance relation (40).

Table 2 Case I: Partially immersed barrier with barrier length *a*/*h*=0.2 and *ε*=0.1

Kh	R_0	R_1	$\|R\|$	T_0	$\|T\|$	$\|R\|^2+\|T\|^2$
0.05	0.000+0.006 2i	−0.014 0+0.000 1i	0.006 3	1.000 0−0.006 2i	1.000 0	1.000 0
0.10	0.000 1+0.009 1i	−0.016 1+0.000 1i	0.009 2	0.999 9−0.009 1i	0.999 9	1.000 0
0.30	0.000 4+0.018 7i	−0.079 4+0.001 5i	0.020 3	0.999 6−0.018 7i	0.999 8	0.999 9
0.50	0.001 2+0.030 5i	−0.223 2+0.006 9i	0.037 6	0.998 8−0.030 5i	0.999 3	1.000 0
0.80	0.002 1+0.044 9i	0.016 6−0.000 7i	0.045 0	0.997 9−0.044 9i	0.998 9	0.999 8
1.00	0.003 8+0.060 4i	−0.021 7+0.001 3i	0.060 6	0.996 2−0.060 4i	0.998 0	0.999 7

Table 3 Case II: Bottom standing barrier with gap *b*/*h*=0.6 and *ε*=0.1

Kh	R_0	R_1	$\|R\|$	T_0	$\|T\|$	$\|R\|^2+\|T\|^2$
0.05	0.005+0.0224i	−0.013 5+0.000 3i	0.022 5	0.999 5−0.022 4i	0.999 7	1.000 0
0.10	0.001 0+0.031 3i	−0.026 2+0.000 8i	0.031 4	0.999 0−0.031 3i	0.999 5	1.000 0
0.30	0.002 7+0.051 3i	−0.060 0+0.003 1i	0.051 7	0.997 3−0.051 3i	0.998 6	0.999 9
0.50	0.003 9+0.061 9i	−0.069 0+0.004 3i	0.062 4	0.996 1−0.061 9i	0.998 0	0.999 9
0.80	0.004 7+0.068 2i	−0.050 1+0.003 4i	0.068 5	0.995 3−0.068 2i	0.997 6	0.999 9
1.00	0.004 8+0.068 6i	−0.028 8+0.002 0i	0.068 9	0.995 2−0.068 6i	0.997 6	0.999 9

Table 4 Case III: Barrier with a gap for *e*/*h*=0.1, *d*/*h*=0.3 $e/h = 0.1$, and *ε*=0.1

Kh	R_0	R_1	$\|R\|$	T_0	$\|T\|$	$\|R\|^2+\|T\|^2$
0.05	0.002 2+0.046 8i	−0.016 0+0.000 8i	0.046 8	0.997 8−0.046 8i	0.998 9	1.000 0
0.10	0.004 4+0.065 8i	0.007 6−0.000 5i	0.065 9	0.995 6−0.065 8i	0.997 8	1.000 0
0.30	0.013 3+0.113 9i	0.093 4−0.010 8i	0.115 0	0.986 7−0.113 9i	0.993 3	0.999 8
0.50	0.021 7+0.144 7i	−0.259 2+0.038 5i	0.148 6	0.978 3−0.144 7i	0.988 9	1.000 0
0.80	0.031 5+0.173 9i	0.005 6−0.001 0i	0.176 7	0.968 5−0.173 9i	0.984 0	0.999 6
1.00	0.037 8+0.190 3i	−0.038 4+0.007 6i	0.194 0	0.962 2−0.190 3i	0.980 8	0.999 6

Table 5 Case I: Partially immersed barrier with barrier length *a*/*h*=0.2 and *ε*=0.001

Kh	R_0	R_1	$\|R\|$	T_0	$\|T\|$	$\|R\|^2+\|T\|^2$
0.05	0.000+0.005 9i	−0.016 6+0.000 1i	0.005 9	1.000 0−0.005 9i	1.000 0	1.000 0
0.10	0.000 1+0.008 7i	−0.030 2+0.000 3i	0.008 7	0.999 9−0.008 7i	1.000 0	1.000 0
0.30	0.000 3+0.017 8i	−0.065 8+0.001 2i	0.017 8	0.999 7−0.017 8i	0.999 8	0.999 9
0.50	0.000 8+0.027 4i	−0.076 4+0.002 1i	0.027 4	0.999 2−0.027 4i	0.999 6	0.999 9
0.80	0.002 1+0.044 9i	−0.061 8+0.002 8i	0.044 9	0.997 9−0.044 9i	0.998 9	0.999 8
1.00	0.003 8+0.060 4i	−0.044 7+0.002 7i	0.060 5	0.996 2−0.060 4i	0.998 0	0.999 7

8 Conclusions

Perturbation analysis is used to solve the mixed BVP arising in the study of scattering of oblique incident waves by irregular bottom and thin vertical barrier. The zeroth order reflection and transmission coefficients are obtained by employing the eigenfunction expansion method leading to an over-determined system of equations which is solved with the help of least-squares approximation and thereby employing the QR-factorization. Using the Green's integral theorem, the first order reflection and transmission coefficients are obtained in terms of integrals containing the shape function $c(x)$ and the solution of the problem involving scattering of surface waves by a vertical barrier over uniform finite depth water. The effect of barrier length, amplitude of the ripples, the number of ripples and the angle of incidence on the first order reflection coefficient is demonstrated through different figures. It is found that the global maximum value of the first order reflection coefficient is increasing with increasing the number of ripples and it is also increasing with increasing the barrier length and the amplitude of the ripples. But the reflection coefficient is decreasing with increasing the angle of incidence and the gap. All these numerical results agree with the physical intuitions of the problem. The energy balance relation is derived and used to check the correctness of the numerical results involving the physical quantities obtained by the present method.

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