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Construction of Wave-free Potentials and Multipoles in a Two-layer Fluid Having Free-surface Boundary Condition with Higher-order Derivatives

Dilip Das*

Department of Mathematics, Shibpur Dinobundhoo Institution (College), Shibpur, Howrah-711102, West Bengal, India

Abstract: There is a large class of problems in the field of fluid structure interaction where higher-order boundary conditions arise for a second-order partial differential equation. Various methods are being used to tackle these kind of mixed boundary-value problems associated with the Laplace's equation (or Helmholtz equation) arising in the study of waves propagating through solids or fluids. One of the widely used methods in wave structure interaction is the multipole expansion method. This expansion involves a general combination of a regular wave, a wave source, a wave dipole and a regular wave-free part. The wave-free part can be further expanded in terms of wave-free multipoles which are termed as wave-free potentials. These are singular solutions of Laplace's equation or two-dimensional Helmholz equation. Construction of these wave-free potentials and multipoles are presented here in a systematic manner for a number of situations such as two-dimensional non-oblique and oblique waves, three dimensional waves in two-layer fluid with free surface condition with higher order partial derivative are considered. In particular, these are obtained taking into account of the effect of the presence of surface tension at the free surface and also in the presence of an ice-cover modelled as a thin elastic plate. Also for limiting case, it can be shown that the multipoles and wave-free potential functions go over to the single layer multipoles and wave-free potential.

Keywords: two-layer fluid, wave-free potentials, Laplace's equation, modified Helmholtz equations, higher order boundary conditions; multipoles

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1 Introduction

The motion of a body of any geometrical configuration, floating on the surface of water, is investigated in the literature assuming linearized theory of water waves. The problem of heaving motion of a long, horizontal circular cylinder on the surface of water was investigated by Ursell (1949) using the method of multipole expansion of the time-harmonic stream function. The corresponding velocity potential also has a similar expansion. In fact, for an infinitely long horizontal cylinder of arbitrary cross section floating on the surface of water, the potential function in general can be expressed in terms of a regular wave, a wave source, a dipole and wave-free potentials (Ursell, 1968; Athanassonlis, 1984). The wave-free potentials are singular at some point and tend to zero rapidly at infinity. Obviously these satisfy the free-surface condition. Two and three dimensional problem of multipole expansions in the theory of surface waves in infinite deep water and also in water of uniform finite depth water has been given by Thorne (1953). Expansions in terms of the wave source and an infinite set of wave-free potentials were introduced for the three-dimensional problem involving a floating sphere half-immersed and making periodic heaving oscillations by Havelock (1955). Ursell (1961a; 1961b), Bolton and Ursell (1973), Mandal and Goswami (1984) considered problems where the potential functions is expansion in terms of wave sources and wave-free potentials. Taylor and Hu (1991) described expansion of the velocity potential for two and three dimensional wave diffraction and radiation problems. Linton and McIver (2001) briefly described the construction of wave free potentials in the case of water of infinite and finite depth water with a free surface.

There is a large class of problems in the field of fluid structure interaction where higher-order boundary conditions arise for a second-order partial differential equation. Various methods are being used to tackle these kind of mixed boundary-value problems (BVP) associated with the Laplace equation (or Helmholtz equation) arising in the study of waves propagating through solids or fluids. One of the widely used methods in wave structure interaction is the method multipole expansion. In most of the wave-structure interaction problems, the governing equation is either the Laplace or the Helmholtz equation and, thus, the features of the orthogonal relation mainly depend upon the nature of the bottom and upper surface boundary conditions. Higher-order boundary conditions occur frequently in fluid-structure interaction problems when we deal with very large floating structures (VLFS). Evans and Porter (2003) analysed the oblique wave scattering caused by a narrow crack in ice sheets floating on water of finite depth with the eigenfunction expansion method. Chakrabarti (2000) analysed the problem of scattering of surface water waves by the edge of an ice cover and obtained the explicit

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***Corresponding author Email:** dilipdas99@gmail.com

solution with a singular, Carleman-type integral equation. Das and Mandal (2006; 2009) analysed the problem of water wave scattring by a circular cylinder with an ice-cover. Das and Mandal (2008) also studied the water wave radiation by a sphere submerged in water with an ice-cover. Das and Mandal (2010a), Mandal and Das (2010) presented in a systematic manner the construction of wave free potentials for two- dimensional deep water and finite depth water with free surface, or corresponding the effect of the surface tension at the free surface and also in water with an ice-cover. Dhillon and Mandal (2013) also presented the construction of a wave free potential for three dimensional deep water as well as finite depth water.

The multipoles and wave-free potentials have been described for single layer fluids, both in two and three dimensions and for infinite as well as finite depth water. More recently, however, interest has been extended to bodies which are floating, submerged or partially immersed in two-layer fluids, each fluid having a different density. The wave motion in a two-layer fluid has gained importance due to plans to construct under water pipe bridge across the Norwegian fjords. A fjord consists of a layer of fresh water on the top of a deep layer of salt water. For tow-dimension motions, Kassem (1982) presented the multipole expansions for two superposed fluids, each of finite depth. Linton and McIver (1995), Linton and Cadby (2002) have constructed a set of multipoles in the theory of surface waves in two-layer fluid with free surface for normal and oblique incident wave trains. Three-dimensional multipoles in two-layer fluid with free surface have been given by Cadby and Linton (2000). Das and Mandal (2007; 2010b) constructed the multipole potentials in their problems for two-dimension and three-dimension in two-layer fluid with an ice-cover.

In the literature, two-layer fluid water wave problems have not been studied for wave-free potential function construction. However, for the various classes of water wave problems in two-layer fluid many researchers may use the wave-free potentials in the mathematical analysis. For circular cylinder of arbitrary cross section floating on the surface of a two-layer fluid or half-immersed circular cylinder in a two-layer fluid, the potential function in general can be expressed in terms of a regular wave, wave-free potential *etc*. Also in winter, a fjord is covered by a layer of ice, so that we have a cylindrical pipe bridge submerged below an ice-cover. Now if we consider the problem of partially or half-immersed circular cylinder in a two-layer fluid with an ice-cover, then potential functions may be expressed in terms of the regular potential as well as wave-free potential function. Thus it will study the problem of construction of wave-free potential in a two-layer fluid. Also in these problems (both single layer and two-layer) the higher-order boundary condition involves third order partial derivative (surface tension) and fifth order partial derivation (ice-cover). However, the boundary value problem involving higher-order boundary conditions more than fifth order partial derivative (Manam *et al*., 2006; Das *et al*., 2008) have not been extensively studied with a view to establish the multipole potentials and also wave-free potentials.

In this paper construction of wave-free potentials and multipoles are presented in a systematic manner. The cases of two-dimensional non-oblique and oblique waves in two-layer fluid with free surface condition with higher order partial derivative are considered. Also the cases of three-dimensional waves in two-layer fluid with free surface with higher order partial derivative are considered. When the higher order partial derivative reduces to first order (free surface) or fifth order partial derivative (ice-cover), multipoles exactly coincide with the multipoles for two-layer fluid with free surface (Linton and McIver, 1995; Linton and Cadby, 2002; Cadby and Linton, 2000), or for two-layer fluid with ice-cover (Das and Mandal, 2007; 2010b).

2 Multipoles and wave-free potentials for non-oblique waves

In a two-layer fluid, both the upper and lower fluids are assumed to be homogeneous, incompressible and inviscid. Let ρ^{I} be the density of the upper fluid and ρ^{II} ($>\rho^{\mathrm{I}}$) be the same for the lower fluid. Let the lower fluid extend infinitely downwards while the upper one has a finite height h above the mean interface. Let y-axis points vertically upwards from the undisturbed interface y=0. Thus the upper layer occupies the region $0<y<h$ while the lower layer occupies the region $y<0$. Under the usual assumption of linear theory and irrotational two-dimensional motion, velocity potentials $\mathrm{Re}\{\varphi^{\mathrm{I,II}}(x,y)\mathrm{e}^{-\mathrm{i}\omega t}\}$, ω being angular velocity, describing the fluid motion in the upper and lower layers exist. For a general BVP, $\varphi^{\mathrm{I,II}}$ satisfy

$$\nabla^2\varphi^{\mathrm{I}}=0,\quad 0<y<h \tag{1}$$

$$\nabla^2\varphi^{\mathrm{II}}=0,\quad y<0 \tag{2}$$

On the upper surface having the mean position $y=h$, φ^{I} satisfies the free-surface condition with higher-order derivatives of the form (Landau and Lifshitz, 1959):

$$\left(D\frac{\partial^4}{\partial x^4}+1-\varepsilon K\right)\varphi^{\mathrm{I}}_y+K\varphi^{\mathrm{I}}=0 \quad \text{on} \quad y=h \tag{3}$$

If the free-surface has an ice-cover modelled as a thin elastic plate, where $D=Eh_0^3/12(1-\nu^2)\rho g$, $\varepsilon=\rho_0 h_0/\rho$, ρ_0 is the density of ice, ρ is density of water, h_0 is the small thickness of ice-cover, E, ν are the Young's modulus and Poission's ratio of the ice and $K=\omega^2/g$, g being the acceleration due to gravity. A generalization of (2) for more higher-order derivatives has been introduced by Manam *et al*. (2006) and has the form:

$$\mathcal{L}\varphi_y+K\varphi=0 \qquad \text{on} \quad y=0 \tag{4}$$

where $\mathcal{L}$ is a linear differential operator of the form:

$$\mathcal{L} = \sum_{m=0}^{m_0} c_m \frac{\partial^{2m}}{\partial x^{2m}} \tag{5}$$

where c_m (m=0, 1,…, m_0) are known constants. Keeping in mind various physical problems involving fluid structure interaction, only the even order partial derivatives in x are considered in the differential operator $\mathcal{L}$.

The linearised boundary conditions at the interface y=0 are

$$\varphi_y^{\mathrm{I}} = K\varphi_y^{\mathrm{II}} \quad \text{on} \quad y=0 \tag{6}$$

$$\rho(\varphi_y^{\mathrm{I}} - K\varphi^{\mathrm{I}}) = \varphi_y^{\mathrm{II}} - K\varphi^{\mathrm{II}} \quad \text{on} \quad y=0 \tag{7}$$

where $\rho = \rho^{\mathrm{I}} / \rho^{\mathrm{II}} (<1)$ and the bottom condition is given by

$$\nabla\varphi^{\mathrm{II}} \to 0 \quad \text{as} \quad y \to -\infty \tag{8}$$

2.1 Singularities in the lower layer

We first consider solutions of Laplace equation in two dimensions (x, y) which are singular at (0, f<0). Polar co-ordinates (r, θ) are defined in the (x, y)-plane by

$$x = r\sin\theta \quad \text{and} \quad y = f - r\cos\theta \tag{9}$$

Now for the case of normal incidence, the solutions of Laplace's equation singular at y=f<0 are $r^{-n}\cos n\theta$ and $r^{-n}\sin n\theta,\ n \geq 1$, and these have the integral representations (Thorne, 1953)

$$\frac{\cos n\theta}{r^n} = \begin{cases} \dfrac{1}{(n-1)!}\displaystyle\int_0^\infty k^{n-1}\mathrm{e}^{-k(y-f)}\cos kx\mathrm{d}k, & y > f \\ \dfrac{(-1)^n}{(n-1)!}\displaystyle\int_0^\infty k^{n-1}\mathrm{e}^{k(y-f)}\cos kx\mathrm{d}k, & y < f \end{cases} \tag{10}$$

$$\frac{\sin n\theta}{r^n} = \begin{cases} \dfrac{1}{(n-1)!}\displaystyle\int_0^\infty k^{n-1}\mathrm{e}^{-k(y-f)}\sin kx\mathrm{d}k, & y > f \\ \dfrac{(-1)^n}{(n-1)!}\displaystyle\int_0^\infty k^{n-1}\mathrm{e}^{k(y-f)}\sin kx\mathrm{d}k, & y < f \end{cases} \tag{11}$$

Let φ_n^s and φ_n^a denote the symmetric and antisymmetric multipoles satisfying (1), (2) except at (0, f) with boundary conditions (4) to (7) and

$$\varphi_n^{\mathrm{II}s} \to \frac{\cos n\theta}{r^n} \quad \text{as} \quad r \to 0 \tag{12}$$

$$\varphi_n^{\mathrm{II}a} \to \frac{\sin n\theta}{r^n} \quad \text{as} \quad r \to 0 \tag{13}$$

Also they represent outgoing waves as $|x| \to \infty$.

The mutipoles are constructed as Linton and McIver (1995)

$$\varphi_n^{\mathrm{I}s} = \frac{(-1)^n}{(n-1)!}\int_0^\infty k^{n-1}\left(A(k)\mathrm{e}^{ky} + B(k)\mathrm{e}^{-ky}\right)\cos kx\mathrm{d}k \tag{14}$$

$$\varphi_n^{\mathrm{II}s} = \frac{\cos n\theta}{r^n} + \frac{(-1)^n}{(n-1)!}\int_0^\infty k^{n-1}C(k)\mathrm{e}^{ky}\cos kx\mathrm{d}k \tag{15}$$

$$\varphi_n^{\mathrm{I}a} = \frac{(-1)^{n+1}}{(n-1)!}\int_0^\infty k^{n-1}\left(A(k)\mathrm{e}^{ky} + B(k)\mathrm{e}^{-ky}\right)\sin kx\mathrm{d}k \tag{16}$$

$$\varphi_n^{\mathrm{II}a} = \frac{\sin n\theta}{r^n} + \frac{(-1)^{n+1}}{(n-1)!}\int_0^\infty k^{n-1}C(k)\mathrm{e}^{ky}\sin kx\mathrm{d}k \tag{17}$$

where $A(k)$, $B(k)$, $C(k)$ are functions of k to be found such that the integrals exist in some sense and satisfy the generalized boundary condition (4) and the interface conditions (5) and (6) and are of outgoing nature at infinity. All the conditions are satisfied if we choose $A(k)$, $B(k)$ and $C(k)$ as

$$A(k) = K\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right) + K\right\}\mathrm{e}^{k(f-h)} / H(k) \tag{18}$$

$$B(k) = K\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right) - K\right\}\mathrm{e}^{k(f+h)} / H(k) \tag{19}$$

$$C(k) = \left[K\rho\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right)\cosh kh - K\sinh kh\right\} - \{(1-\rho)k + K\}\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right)\sinh kh - K\cosh kh\right\}\right] \times \mathrm{e}^{kf} / H(k) \tag{20}$$

where $H(k)$ is given by

$$H(k) = K\rho\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right)\cosh kh - K\sinh kh\right\} - \{k(1-\rho) - K\}\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right)\sinh kh - K\cosh kh\right\} \tag{21}$$

The path of the integration in the integrals in (14) to (17) is indented below the poles at k=k_1 and k=k_2 on the real k-axis to take care of their outgoing behaviour as $|x| \to \infty$ and k_1, k_2 are only two real positive roots of the equation $H(k)$=0 (Das *et al*., 2008).

The far-field forms of the multipoles, in the lower fluid, is given by

$$\varphi_n^{\mathrm{II}s} \sim a_n^{(1)}$$

$$\varphi_n^{\mathrm{II}a} \sim b_n^{(1)}$$

where

$$a_n^{(1)} = \frac{(-1)^n}{(n-1)!}\pi\mathrm{i}(k_1^{n-1}C^{k_1}\mathrm{e}^{\pm\mathrm{i}k_1x + k_1y} + k_2^{n-1}C^{k_2}\mathrm{e}^{\pm\mathrm{i}k_2x + k_2y}) \tag{22}$$

$$b_n^{(1)} = \frac{(-1)^n}{(n-1)!}\pi(k_1^{n-1}C^{k_1}\mathrm{e}^{\pm\mathrm{i}k_1x + k_1y} + k_2^{n-1}C^{k_2}\mathrm{e}^{\pm\mathrm{i}k_2x + k_2y}) \tag{23}$$

as $x \to \pm\infty$. Here C^{k_1}, C^{k_2} are the residues of $C(k)$ at k=k_1 and k=k_2, given by

$$C^{k_j} = \left[K\rho\left\{k_j\left(\sum_0^{m_0}(-1)^m c_m k_j^{2m}\right)\cosh k_j h - K\sinh k_j h\right\} - \{(1-\rho)k_j + K\}\left\{k_j\left(\sum_0^{m_0}(-1)^m c_m k_j^{2m}\right)\sinh k_j h - K\cosh k_j h\right\}\right] \times \mathrm{e}^{k_j f} / H'(k_j), \quad j = 1,2. \tag{24}$$

Using (22) and (23), we find

$$a_{n+2}^{(1)} + \frac{k_1 + k_2}{n+1}a_{n+1}^{(1)} + \frac{k_1 k_2}{(n+1)n}a_n^{(1)} = 0 \tag{25}$$

$$b_{n+2}^{(1)} + \frac{k_1+k_2}{n+1} b_{n+1}^{(1)} + \frac{k_1 k_2}{(n+1)n} b_n^{(1)} = 0 \tag{26}$$

The combinations $\varphi_{n+2}^{\mathrm{II}s,a} + \frac{k_1+k_2}{n+1}\varphi_{n+1}^{\mathrm{II}s,a} + \frac{k_1k_2}{(n+1)n}\varphi_n^{\mathrm{II}s,a}$ do not contribute anything as $|x|\to\infty$ so that they are wave-free. Now using the representations (25) and (26) it can be shown that

$$\begin{aligned}&\varphi_{n+2}^{\mathrm{II}s} + \frac{k_1+k_2}{n+1}\varphi_{n+1}^{\mathrm{II}s} + \frac{k_1k_2}{(n+1)n}\varphi_n^{\mathrm{II}s} = \\ &\frac{\cos(n+2)\theta}{r^{n+2}} + \frac{k_1+k_2}{n+1}\frac{\cos(n+1)\theta}{r^{n+1}} + \frac{k_1k_2}{n(n+1)}\frac{\cos n\theta}{r^n} + \\ &\frac{(-1)^n}{(n+1)!}\int_0^\infty (k-k_1)(k-k_2)k^{n-1}C(k)\mathrm{e}^{ky}\cos kx\,\mathrm{d}k\end{aligned} \tag{27}$$

and

$$\begin{aligned}&\varphi_{n+2}^{\mathrm{II}a} + \frac{k_1+k_2}{n+1}\varphi_{n+1}^{\mathrm{II}a} + \frac{k_1k_2}{(n+1)n}\varphi_n^{\mathrm{II}a} = \\ &\frac{\sin(n+2)\theta}{r^{n+2}} + \frac{k_1+k_2}{n+1}\frac{\sin(n+1)\theta}{r^{n+1}} + \frac{k_1k_2}{n(n+1)}\frac{\sin n\theta}{r^n} + \\ &\frac{(-1)^{n+1}}{(n+1)!}\int_0^\infty (k-k_1)(k-k_2)k^{n-1}C(k)\mathrm{e}^{ky}\sin kx\,\mathrm{d}k\end{aligned} \tag{28}$$

Letting $f\to 0$ in (27) and (28) we obtain the symmetric and antisymmetric wave-free potentials with singularity near the interface between two-layer and are given by

$$\begin{aligned}\chi_m^{\mathrm{II}s} = &\frac{\cos(m+2)\theta}{r^{m+2}} + \frac{k_1+k_2}{m+1}\frac{\cos(m+1)\theta}{r^{m+1}} + \frac{k_1+k_2}{m+1}\frac{\cos m\theta}{r^m} + \\ &\frac{(-1)^m}{(m+1)!}\int_0^\infty (k-k_1)(k-k_2)k^{m-1}C^*(k)\mathrm{e}^{ky}\cos kx\,\mathrm{d}k\end{aligned} \tag{29}$$

and

$$\begin{aligned}\chi_m^{\mathrm{II}a} = &\frac{\sin(m+2)\theta}{r^{m+2}} + \frac{k_1+k_2}{m+1}\frac{\sin(m+1)\theta}{r^{m+1}} + \frac{k_1k_2}{m(m+1)}\frac{\sin m\theta}{r^m} + \\ &\frac{(-1)^{m+1}}{(m+1)!}\int_0^\infty (k-k_1)(k-k_2)k^{m-1}C^*(k)\mathrm{e}^{ky}\sin kx\,\mathrm{d}k\end{aligned} \tag{30}$$

where $C^*(k)$ is the limiting value of $C(k)$ when $f\to 0$.

2.2 Singularities in the upper layer

To develop multipoles singular at $y=f>0$ and polar co-ordinates are again defined via (9). The solutions of Laplace's equation singular at $y=f>0$ are $r^{-n}\cos n\theta$ and $r^{-n}\sin n\theta$, $n\ge 1$, and let φ_n^s and φ_n^a denote the symmetric and antisymmetric multipoles satisfying (1), (2) except at $(0, f)$ with boundary conditions (4) to (7) and

$$\varphi_n^{\mathrm{I}s} \to \frac{\cos n\theta}{r^n} \quad \text{as} \quad r\to 0 \tag{31}$$

$$\varphi_n^{\mathrm{I}a} \to \frac{\sin n\theta}{r^n} \quad \text{as} \quad r\to 0 \tag{32}$$

Also they represent outgoing waves as $|x|\to\infty$.

The mutipoles are constructed as (Linton and McIver, 1995)

$$\begin{aligned}\varphi_n^{\mathrm{I}s} = &\frac{\cos n\theta}{r^n} + \\ &\frac{1}{(n-1)!}\int_0^\infty \left(A_n^{(0)}(k)\mathrm{e}^{ky} + B_n^{(0)}(k)\mathrm{e}^{-ky}\right)k^{n-1}\cos kx\,\mathrm{d}k\end{aligned} \tag{33}$$

$$\varphi_n^{\mathrm{II}s} = \frac{1}{(n-1)!}\int_0^\infty k^{n-1}C_n^{(0)}(k)\mathrm{e}^{ky}\cos kx\,\mathrm{d}k \tag{34}$$

$$\begin{aligned}\varphi_n^{\mathrm{I}a} = &\frac{\sin n\theta}{r^n} + \\ &\frac{1}{(n-1)!}\int_0^\infty k^{n-1}\left(A_n^{(1)}(k)\mathrm{e}^{ky} + B_n^{(1)}(k)\mathrm{e}^{-ky}\right)\sin kx\,\mathrm{d}k\end{aligned} \tag{35}$$

$$\varphi_n^{\mathrm{II}a} = \frac{1}{(n-1)!}\int_0^\infty k^{n-1}C_n^{(1)}(k)\mathrm{e}^{ky}\sin kx\,\mathrm{d}k \tag{36}$$

where

$$\begin{aligned}A_n^{(j)}(k) = &\frac{1}{2}\left[\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right) + K\right\}\mathrm{e}^{-kh}\times\right. \\ &\left\{(-1)^{n+j+1}((1-\rho)k-(1+\rho)K)\mathrm{e}^{kf} - \right. \\ &\left.\left.(1-\rho)(k-K)\mathrm{e}^{-kf}\right\}\right]/H(k),\quad j=0,1\end{aligned} \tag{37}$$

$$\begin{aligned}B_n^{(j)}(k) = &\frac{1}{2}\left[(-1)^{n+j+1}\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right) + K\right\}\times\right. \\ &((1-\rho)(k-K)\mathrm{e}^{k(f-h)} - (1-\rho)(k-K)\times \\ &\left.\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right) - K\right\}\mathrm{e}^{-k(f-h)}\right]/H(k)\qquad j=1,2\end{aligned} \tag{38}$$

$$\begin{aligned}C_n^{(j)}(k) = &-\rho K\left[(-1)^{n+j+1}\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right) + K\right\}\mathrm{e}^{k(f-h)} - \right. \\ &\left.\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right) - K\right\}\mathrm{e}^{-k(f-h)}\right]/H(k)\quad j=0,1\end{aligned} \tag{39}$$

The path of the integration in the integrals in (33) to (36) is indented below the poles at $k=k_1$ and $k=k_2$ on the real k-axis to take care of their outgoing behaviour as $|x|\to\infty$.

The far-field forms of the multipoles, in the upper fluid, is given by

$$\varphi_n^{\mathrm{I}s} \sim a_n^{(2)}$$

$$\varphi_n^{\mathrm{I}a} \sim b_n^{(2)}$$

where

$$\begin{aligned}a_n^{(2)} = &\frac{1}{(n-1)!}\pi\mathrm{i}[k_1^{n-1}(A_n^{(0)}(k_1)\mathrm{e}^{k_1y} + B_n^{(0)}(k_1)\mathrm{e}^{-k_1y})\mathrm{e}^{\pm\mathrm{i}k_1x} + \\ &k_2^{n-1}(A_n^{(0)}(k_2)\mathrm{e}^{k_2y} + B_n^{(0)}(k_2)\mathrm{e}^{-k_2y})\mathrm{e}^{\pm\mathrm{i}k_2x}]\end{aligned} \tag{40}$$

$$\begin{aligned}b_n^{(2)} = &\frac{1}{(n-1)!}\pi[k_1^{n-1}(A_n^{(1)}(k_1)\mathrm{e}^{k_1y} + B_n^{(1)}(k_1)\mathrm{e}^{-k_1y})\mathrm{e}^{\pm\mathrm{i}k_1x} + \\ &k_2^{n-1}(A_n^{(1)}(k_2)\mathrm{e}^{k_2y} + B_n^{(1)}(k_2)\mathrm{e}^{-k_2y})\mathrm{e}^{\pm\mathrm{i}k_2x}]\end{aligned} \tag{41}$$

as $x\to\pm\infty$. Here $A_n^{(j)}(k_1)$, $A_n^{(j)}(k_2)$, and $B_n^{(j)}(k_1)$, $B_n^{(j)}(k_2)$ are the residues of $A_n^{(j)}(k)$ and $B_n^{(j)}(k)$ respectively at $k=k_1$ and $k=k_2$, given by

$$\begin{aligned}A_n^{(j)}(k_s) = (-1)^{n+j+1}M_1(k_s) + M_2(k_s) \\ j=1,2,\quad s=1,2\end{aligned} \tag{42}$$

$$B_n^{(j)}(k_s)=(-1)^{n+j+1}N_1(k_s)+N_2(k_s)\qquad j=1,2,\qquad s=1,2 \tag{43}$$

where

$$M_1(k_s)=\frac{1}{2}\left[\left\{k_s\left(\sum_0^{m_0}(-1)^m c_m k_s^{2m}\right)+K\right\}\mathrm{e}^{-k_s h}\times\left\{(-1)^{n+j+1}((1-\rho)k_s-(1+\rho)K)\mathrm{e}^{k_s f}\right\}\right]/H'(k_s)$$

$$M_2(k_s)=-\frac{1}{2}\mathrm{e}^{-k_s h}\left\{(1-\rho)(k_s-K)\mathrm{e}^{-k_s f}\right\}\times\left[\left\{k_s\left(\sum_0^{m_0}(-1)^m c_m k_s^{2m}\right)+K\right\}\right]/H'(k_s)$$

$$N_1(k_s)=\frac{1}{2}(1-\rho)(k_s-K)\mathrm{e}^{k_s(f-h)}\times\left[(-1)^{n+j+1}\left\{k_s\left(\sum_0^{m_0}(-1)^m c_m k_s^{2m}\right)+K\right\}\right]/H'(k_s)$$

$$N_2(k_s)=-\frac{1}{2}\left\{k_s\left(\sum_0^{m_0}(-1)^m c_m k_s^{2m}\right)-K\right\}\times\left[(1-\rho)(k_s-K)\mathrm{e}^{-k_s(f-h)}\right]/H'(k_s),\qquad s=1,2$$

Using (40) and (41), we find

$$a_{n+2}^{(2)}+\frac{k_1+k_2}{n+1}a_{n+1}^{(2)}+\frac{k_1k_2}{(n+1)n}a_n^{(2)}-\frac{2\pi\mathrm{i}}{(n+1)!}(k_1+k_2)\times\left\{(M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})k_1^n\mathrm{e}^{\pm ik_1x}+(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})k_2^n\mathrm{e}^{\pm ik_2x}\right\}=0 \tag{44}$$

$$b_{n+2}^{(2)}+\frac{k_1+k_2}{n+1}b_{n+1}^{(2)}+\frac{k_1k_2}{(n+1)n}b_n^{(2)}-\frac{2\pi}{(n+1)!}(k_1+k_2)\times\left\{(M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})k_1^n\mathrm{e}^{\pm ik_1x}+(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})k_2^n\mathrm{e}^{\pm ik_2x}\right\}=0 \tag{45}$$

Now using the representations (44) and (45) it can be shown that

$$\begin{aligned}&\varphi_{n+2}^{\mathrm{Is}}+\frac{k_1+k_2}{n+1}\varphi_{n+1}^{\mathrm{Is}}+\frac{k_1k_2}{(n+1)n}\varphi_n^{\mathrm{Is}}-\frac{2\pi\mathrm{i}}{(n+1)!}(k_1+k_2)\times\\&\left\{(M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})k_1^n\mathrm{e}^{\pm\mathrm{i}k_1x}+(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})k_2^n\mathrm{e}^{\pm\mathrm{i}k_2x}\right\}=\\&\frac{\cos(n+2)\theta}{r^{n+2}}+\frac{k_1+k_2}{n+1}\frac{\cos(n+1)\theta}{r^{n+1}}+\frac{k_1k_2}{n(n+1)}\frac{\cos n\theta}{r^n}+\\&\frac{1}{(n+1)!}\int_0^\infty[(-1)^{n+1}(M_1(k)\mathrm{e}^{ky}+N_1(k)\mathrm{e}^{-ky})]\times(k-k_1)(k-k_2)k^{n-1}\cos kx\mathrm{d}k+\\&\frac{1}{(n+1)!}\int_0^\infty k^{n-1}[(M_2(k)\mathrm{e}^{ky}+N_2(k)\mathrm{e}^{-ky})]\times(k+k_1)(k+k_2)\cos kx\mathrm{d}k-\\&\frac{2\pi\mathrm{i}}{(n+1)!}(k_1+k_2)\left\{(M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})k_1^n\mathrm{e}^{\pm ik_1x}+(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})k_2^n\mathrm{e}^{\pm ik_2x}\right\}\end{aligned} \tag{46}$$

and

$$\begin{aligned}&\varphi_{n+2}^{\mathrm{Ia}}+\frac{k_1+k_2}{n+1}\varphi_{n+1}^{\mathrm{Ia}}+\frac{k_1k_2}{(n+1)n}\varphi_n^{\mathrm{Ia}}-\frac{2\pi}{(n+1)!}(k_1+k_2)\times\\&\left\{(M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})k_1^n\mathrm{e}^{\pm\mathrm{i}k_1x}+(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})k_2^n\mathrm{e}^{\pm\mathrm{i}k_2x}\right\}=\\&\frac{\sin(n+2)\theta}{r^{n+2}}+\frac{k_1+k_2}{n+1}\frac{\sin(n+1)\theta}{r^{n+1}}+\frac{k_1k_2}{n(n+1)}\frac{\sin n\theta}{r^n}+\\&\frac{1}{(n+1)!}\int_0^\infty k^{n-1}[(-1)^{n+2}(M_1(k)\mathrm{e}^{ky}+N_1(k)\mathrm{e}^{-ky})]\times(k-k_1)(k-k_2)\sin kx\mathrm{d}k+\\&\frac{1}{(n+1)!}\int_0^\infty k^{n-1}[(M_2(k)\mathrm{e}^{ky}+N_2(k)\mathrm{e}^{-ky})]\times(k+k_1)(k+k_2)\sin kx\mathrm{d}k-\\&\frac{2\pi}{(n+1)!}(k_1+k_2)\left\{(M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})k_1^n\mathrm{e}^{\pm ik_1x}+(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})k_2^n\mathrm{e}^{\pm ik_2x}\right\}\end{aligned} \tag{47}$$

where

$$M_1(k)=\frac{1}{2}\left[\mathrm{e}^{-kh}\left\{(-1)^{n+j+1}((1-\rho)k-(1+\rho)K)\mathrm{e}^{kf}\right\}\right]\times\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right)+K\right\}/H(k)$$

$$M_2(k)=-\frac{1}{2}\left[\mathrm{e}^{-kh}\left\{(1-\rho)(k-K)\mathrm{e}^{-kf}\right\}\right]\times\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right)+K\right\}/H(k)$$

$$N_1(k)=\frac{1}{2}\left[(-1)^{n+j+1}(1-\rho)(k-K)\mathrm{e}^{k(f-h)}\right]\times\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right)+K\right\}/H(k)$$

$$N_2(k)=-\frac{1}{2}\left[(1-\rho)(k-K)\mathrm{e}^{-k(f-h)}\right]\times\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right)-K\right\}/H(k)$$

The last contour integrals in (46) and (47) can be written in the form:

$$\begin{aligned}&\frac{1}{(n+1)!}\int_0^\infty k^{n-1}[(M_2(k)\mathrm{e}^{ky}+N_2(k)\mathrm{e}^{-ky})]\times(k+k_1)(k+k_2)\cos kx\mathrm{d}k+\\&\frac{2\pi\mathrm{i}}{(n+1)!}(k_1+k_2)\left\{(M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})k_1^n\mathrm{e}^{\pm ik_1x}+(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})k_2^n\mathrm{e}^{\pm ik_2x}\right\}\end{aligned} \tag{48}$$

and

$$\frac{1}{(n+1)!}\int_0^\infty k^{n-1}[(M_2(k)\mathrm{e}^{ky}+N_2(k)\mathrm{e}^{-ky})]\times \\ (k+k_1)(k+k_2)\sin kx\mathrm{d}k+ \\ \frac{2\pi}{(n+1)!}(k_1+k_2)\{(M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})k_1^n\mathrm{e}^{\pm\mathrm{i}k_1x}+ \\ (M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})k_2^n\mathrm{e}^{\pm\mathrm{i}k_2x}\} \tag{49}$$

the integrals being in the sense of Cauchy principal value.

Thus (46) and (47) reduce to

$$\varphi_{n+2}^{\mathrm{Is}}+\frac{k_1+k_2}{n+1}\varphi_{n+1}^{\mathrm{Is}}+\frac{k_1k_2}{(n+1)n}\varphi_n^{\mathrm{Is}}-\frac{2\pi\mathrm{i}}{(n+1)!}(k_1+k_2)\times \\ \{(M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})k_1^n\mathrm{e}^{\pm\mathrm{i}k_1x}+ \\ (M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})k_2^n\mathrm{e}^{\pm\mathrm{i}k_2x}\}= \\ \frac{\cos(n+2)\theta}{r^{n+2}}+\frac{k_1+k_2}{n+1}\frac{\cos(n+1)\theta}{r^{n+1}}+\frac{k_1k_2}{n(n+1)}\frac{\cos n\theta}{r^n}+ \\ \frac{1}{(n+1)!}\int_0^\infty[(-1)^{n+1}(M_1(k)\mathrm{e}^{ky}+N_1(k)\mathrm{e}^{-ky})]\times \\ (k-k_1)(k-k_2)k^{n-1}\cos kx\mathrm{d}k+ \\ \frac{1}{(n+1)!}\int_0^\infty k^{n-1}[(M_2(k)\mathrm{e}^{ky}+N_2(k)\mathrm{e}^{-ky})]\times \\ (k+k_1)(k+k_2)\cos kx\mathrm{d}k \tag{50}$$

and

$$\varphi_{n+2}^{\mathrm{Ia}}+\frac{k_1+k_2}{n+1}\varphi_{n+1}^{\mathrm{Ia}}+\frac{k_1k_2}{(n+1)n}\varphi_n^{\mathrm{Ia}}- \\ \{(M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})k_1^n\mathrm{e}^{\pm\mathrm{i}k_1x}+ \\ (M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})k_2^n\mathrm{e}^{\pm\mathrm{i}k_2x}\}= \\ \frac{\sin(n+2)\theta}{r^{n+2}}+\frac{k_1+k_2}{n+1}\frac{\sin(n+1)\theta}{r^{n+1}}+\frac{k_1k_2}{n(n+1)}\frac{\sin n\theta}{r^n}+ \\ \frac{1}{(n+1)!}\int_0^\infty(k-k_1)(k-k_2)k^{n-1}\times \\ [(-1)^{n+2}(M_1(k)\mathrm{e}^{ky}+N_1(k)\mathrm{e}^{-ky})]\sin kx\mathrm{d}k+ \\ \frac{1}{(n+1)!}\int_0^\infty(k+k_1)(k+k_2)k^{n-1}\times \\ [(M_2(k)\mathrm{e}^{ky}+N_2(k)\mathrm{e}^{-ky})]\sin kx\mathrm{d}k \tag{51}$$

the last integrals of (50) and (51) being in the sense of Cauchy principal value. Letting $f\to h$ in (50) and (51) we obtain the symmetric and antisymmetric wave-free potentials with singularity in the upper surface of the two-layer and are given by

$$\chi_m^{\mathrm{Is}}=\frac{\cos(m+2)\theta}{r^{m+2}}+\frac{k_1+k_2}{m+1}\frac{\cos(m+1)\theta}{r^{m+1}}+\frac{k_1k_2}{m(m+1)}\frac{\cos m\theta}{r^m}+ \\ \frac{1}{(m+1)!}\int_0^\infty k^{m-1}[(-1)^{m+1}(M_1^*(k)\mathrm{e}^{ky}+N_1^*(k)\mathrm{e}^{-ky})]\times \\ (k-k_1)(k-k_2)\cos kx\mathrm{d}k+ \\ \frac{1}{(m+1)!}\int_0^\infty k^{m-1}[(M_2^*(k)\mathrm{e}^{ky}+N_2^*(k)\mathrm{e}^{-ky})]\times \\ (k+k_1)(k+k_2)\cos kx\mathrm{d}k \tag{52}$$

and

$$\chi_m^{\mathrm{Ia}}=\frac{\sin(m+2)\theta}{r^{m+2}}+\frac{k_1+k_2}{m+1}\frac{\sin(m+1)\theta}{r^{m+1}}+\frac{k_1k_2}{m(m+1)}\frac{\sin m\theta}{r^m}+ \\ \frac{1}{(m+1)!}\int_0^\infty k^{m-1}[(-1)^{m+2}(M_1^*(k)\mathrm{e}^{ky}+N_1^*(k)\mathrm{e}^{-ky})]\times \\ (k-k_1)(k-k_2)\sin kx\mathrm{d}k+ \\ \frac{1}{(m+1)!}\int_0^\infty k^{m-1}[(M_2^*(k)\mathrm{e}^{ky}+N_2^*(k)\mathrm{e}^{-ky})]\times \\ (k+k_1)(k+k_2)\sin kx\mathrm{d}k \tag{53}$$

where $M_j^*(k)$, $N_j^*(k)$, j=1, 2 are the limiting values of $M_j(k)$, $N_j(k)$, j=1, 2 respectively when $f\to h$.

In particular, choose c_0=1, c_i=0, i=1, 2,…, m_0, the BVP becomes the BVP for two-layer fluid with free surface (Linton and McIver, 1995) and the multipoles exactly coincide with those for the case of two-layer fluid with free surface (Linton and McIver, 1995) and the wave-free potentials become the wave-free potentials for two-layer fluid with free surface. Similarly, if choose $c_0=1-\varepsilon K$, c_1=0, $c_2=D$, c_i=0, i=3, 4,…, m_0, then the BVP becomes the BVP for two layer fluid with ice-cover boundary condition (3) (Das and Mandal, 2007) and obtain the corresponding multipoles (Das and Mandal, 2007) and wave-free potentials. If we let c_0=1, c_i=0, i=1, 2,…, m_0 and $\rho\to 0$ in this problem then it can be shown that the multipoles and wave-free potential functions go over to the single layer multipoles evaluated by Thorne (1953) and wave-free potential evaluated by Das and Mandal (2010a). Thus by letting $\rho\to 0$ in the above analysis we recover the results for the single layer fluid.

3 Multipoles and wave-free potentials for oblique waves

Under the usual assumption of linear theory and irrotational two-dimensional motion, velocity potentials $\mathrm{Re}\{\varphi^{\mathrm{I,II}}(x,y)\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\gamma z}\}$, γ is the wave number component along the z-direction, describing the fluid motion in the upper and lower layers exist. For a general BVP, $\varphi^{\mathrm{I,II}}$ satisfy

$$(\nabla^2-\gamma^2)\varphi^{\mathrm{I}}=0,\quad 0<y<h \tag{54}$$

$$(\nabla^2-\gamma^2)\varphi^{\mathrm{II}}=0,\quad y<0 \tag{55}$$

Eqs. (6) and (7) represent the linearized boundary conditions at the interface y=0, while the free-surface condition with higher-order derivatives at $y=h$ is

$$\mathcal{M}\varphi_y^{\mathrm{I}}-K\varphi^{\mathrm{I}}=0 \qquad \text{on}\quad y=h \tag{56}$$

where the differential operator

$$\mathcal{M}=\sum_{m=0}^{m_0}c_m\left(\frac{\partial^2}{\partial x^2}-\gamma_2\right)^m \tag{57}$$

In (57), c_m (m=0, 1,…, m_0) are known constants. Keeping in mind various physical problems involving fluid structure interaction, only the even order partial derivatives in x are considered in the differential operator $\mathcal{M}$ and (8) is the bottom condition of the lower layer.

3.1 Singularities in the lower layer

Let φ_n^s and φ_n^a denote the symmetric and antisymmetric multipoles satisfying (54) and (55) for upper and lower fluid except at (0, f) with boundary conditions (56) and (6) to (8) and represent an outgoing waves at infinity. Near the point (0, f), the behaviours of $\varphi_n^{\mathrm{II}s,a}$ are given by

$$\varphi_n^{\mathrm{II}s} \to K_n(\gamma r)\cos n\theta \quad \text{as} \quad r \to 0 \tag{58}$$

$$\varphi_n^{\mathrm{II}a} \to K_n(\gamma r)\sin n\theta \quad \text{as} \quad r \to 0 \tag{59}$$

where $K_n(z)$ denotes the modified Bessel function of second kind.

The multipoles are constructed as (Linton and Cadby, 2002)

$$\varphi_n^{\mathrm{I}s} = (-1)^n \int_0^\infty \cosh nk \cos(\gamma x \sinh k) \times \left(A(k)\mathrm{e}^{vy} + B(k)\mathrm{e}^{-vy}\right)\mathrm{d}k \tag{60}$$

$$\varphi_n^{\mathrm{II}s} = K_n(\gamma r)\cos n\theta + (-1)^n \int_0^\infty \cosh nk \cos(\gamma x \sinh k) C(k)\mathrm{e}^{vy}\mathrm{d}k \tag{61}$$

$$\varphi_n^{\mathrm{I}a} = (-1)^{n+1} \int_0^\infty \sinh nk \sin(\gamma x \sinh k) \times \left(A(k)\mathrm{e}^{vy} + B(k)\mathrm{e}^{-vy}\right)\mathrm{d}k \tag{62}$$

$$\varphi_n^{\mathrm{II}a} = K_n(\gamma r)\sin n\theta + (-1)^{n+1} \int_0^\infty \sinh nk \sin(\gamma x \sinh k) C(k)\mathrm{e}^{vy}\mathrm{d}k \tag{63}$$

where $v = \gamma\cosh k$ and $A(k)$, $B(k)$, $C(k)$ are functions of k to be found such that the integrals exist in some sense. Also we have (Thorne, 1953)

$$K_n(\gamma r)\cos n\theta = \begin{cases} \int_0^\infty \cosh nk \cos(\gamma x \sinh k)\mathrm{e}^{-(y-f)\cosh k}\mathrm{d}k, & y > f \\ (-1)^n \int_0^\infty \cosh nk \cos(\gamma x \sinh k)\mathrm{e}^{(y-f)\cosh k}\mathrm{d}k, & y < f \end{cases} \tag{64}$$

$$K_n(\gamma r)\sin n\theta = \begin{cases} \int_0^\infty \sinh nk \sin(\gamma x \sinh k)\mathrm{e}^{-(y-f)\cosh k}\mathrm{d}k, & y > f \\ (-1)^n \int_0^\infty \sinh nk \sin(\gamma x \sinh k)\mathrm{e}^{(y-f)\cosh k}\mathrm{d}k, & y < f \end{cases} \tag{65}$$

The functions φ_n^s and φ_n^a are singular solutions of the modified Helmholtz equation and satisfy the generalized free surface condition (56) and the interface conditions (6) and (7) and are of outgoing nature at infinity. Then $A(k)$, $B(k)$ and $C(k)$ have the forms:

$$A(k) = K\left\{v\left(\sum_{m=0}^{m_0}(-1)^m c_m v^{2m}\right) + K\right\}\mathrm{e}^{v(f-h)} / H(v) \tag{66}$$

$$B(k) = K\left\{v\left(\sum_{m=0}^{m_0}(-1)^m c_m v^{2m}\right) - K\right\}\mathrm{e}^{v(f+h)} / H(v) \tag{67}$$

$$C(k) = \left[K\rho\left\{v\left(\sum_{m=0}^{m_0}(-1)^m c_m v^{2m}\right)\cosh vh - K\sinh vh\right\} - \{(1-\rho)v + K\}\left\{v\left(\sum_{m=0}^{m_0}(-1)^m c_m v^{2m}\right)\sinh vh - K\cosh vh\right\}\right] \times \mathrm{e}^{vf} / H(v) \tag{68}$$

The path of the integration in the integrals in (60) to (63) is indented below the poles at $k = \mu_1$ and $k = \mu_2$, where

$$\gamma\cosh\mu_j = k_j, \quad j = 1,2 \tag{69}$$

k_1, k_2 are only two real positive roots of the equation $H(v)$=0.

The far-field forms of the multipoles, in the lower fluid, is given by

$$\varphi_n^{\mathrm{II}s} \sim a_n^{(3)}$$

$$\varphi_n^{\mathrm{II}a} \sim b_n^{(3)}$$

where

$$a_n^{(3)} = (-1)^n \pi \mathrm{i}(C^{\mu_1}\cosh n\mu_1 + C^{\mu_2}\cosh n\mu_2) \tag{70}$$

$$b_n^{(3)} = (-1)^n \pi(C^{\mu_1}\sinh n\mu_1 + C^{\mu_2}\sinh n\mu_2) \tag{71}$$

as $x \to \pm\infty$, where C^{μ_1} and C^{μ_2} are given by

$$C^{\mu_j} = \left[K\rho\left\{k_j\left(\sum_{m=0}^{m_0}(-1)^m c_m k_j^{2m}\right)\cosh k_j h - K\sinh k_j h\right\} - \{(1-\rho)k_j + K\}\left\{k_j\left(\sum_{m=0}^{m_0}(-1)^m c_m k_j^{2m}\right)\sinh k_j h - K\cosh k_j h\right\}\right] \times \mathrm{e}^{\pm\mathrm{i}\beta_j x + k_j y}\mathrm{e}^{k_j f} / \beta_j H'(k_j), \quad j = 1,2 \tag{72}$$

where $\beta_j = (k_j^2 - \gamma^2)^{1/2}$.

Using (70) and (71), we find

$$a_{n+2}^{(3)} + 2\frac{k_1+k_2}{\gamma}\left(a_{n+1}^{(3)} + a_{n-1}^{(3)}\right) + 2\left(1 + 2\frac{k_1k_2}{\gamma}\right)a_n^{(3)} + a_{n-2}^{(3)} = 0 \tag{73}$$

$$b_{n+2}^{(3)} + 2\frac{k_1+k_2}{\gamma}\left(b_{n+1}^{(3)} + b_{n-1}^{(3)}\right) + 2\left(1 + 2\frac{k_1k_2}{\gamma}\right)b_n^{(3)} + b_{n-2}^{(3)} = 0 \tag{74}$$

Thus

$$\begin{aligned} &\varphi_{n+2}^{\mathrm{II}s} + 2\frac{k_1+k_2}{\gamma}\left(\varphi_{n+1}^{\mathrm{II}s} + \varphi_{n-1}^{\mathrm{II}s}\right) + 2\left(1 + 2\frac{k_1k_2}{\gamma}\right)\varphi_n^{\mathrm{II}s} + \varphi_{n-2}^{\mathrm{II}s} = \\ &K_{n+2}(\gamma r)\cos(n+2)\theta + \\ &2\frac{k_1+k_2}{\gamma}(K_{n+1}(\gamma r)\cos(n+1)\theta + K_{n-1}(\gamma r)\cos(n-1)\theta) + \\ &2\left(1 + 2\frac{k_1k_2}{\gamma}\right)K_n(\gamma r)\cos n\theta + K_{n-2}(\gamma r)\cos(n-2)\theta + \\ &(-1)^n \int_0^\infty 4(\cosh k - \cosh\mu_1)(\cosh k - \cosh\mu_2) \times \\ &C(k)\mathrm{e}^{vy}\cosh nk\cos(\gamma x\sinh k)\mathrm{d}k \end{aligned} \tag{75}$$

and

$$\varphi_{n+2}^{\mathrm{II}a}+2\frac{k_1+k_2}{\gamma}\left(\varphi_{n+1}^{\mathrm{II}a}+\varphi_{n-1}^{\mathrm{II}a}\right)+2\left(1+2\frac{k_1k_2}{\gamma}\right)\varphi_n^{\mathrm{II}a}+\varphi_{n-2}^{\mathrm{II}a}= \\ K_{n+2}(\gamma r)\sin(n+2)\theta+ \\ 2\frac{k_1+k_2}{\gamma}(K_{n+1}(\gamma r)\sin(n+1)\theta+K_{n-1}(\gamma r)\sin(n-1)\theta)+ \\ 2\left(1+2\frac{k_1k_2}{\gamma}\right)K_n(\gamma r)\sin n\theta+K_{n-2}(\gamma r)\sin(n-2)\theta+ \\ (-1)^n\int_0^\infty 4(\cosh k-\cosh\mu_1)(\cosh k-\cosh\mu_2)\times \\ C(k)\mathrm{e}^{vy}\sinh nk\sin(\gamma x\sinh k)\mathrm{d}k \tag{76}$$

Letting $f\to 0$ in (75) and (76) we obtain the symmetric and antisymmetric wave-free potentials with singularity near the interface between two layers and are given by

$$\chi_m^{\mathrm{II}s}=K_{m+2}(\gamma r)\cos(m+2)\theta+ \\ 2\frac{k_1+k_2}{\gamma}(K_{m+1}(\gamma r)\cos(m+1)\theta+K_{m-1}(\gamma r)\cos(m-1)\theta)+ \\ 2\left(1+2\frac{k_1k_2}{\gamma}\right)K_m(\gamma r)\cos m\theta+K_{m-2}(\gamma r)\cos(m-2)\theta+ \\ (-1)^m\int_0^\infty 4(\cosh k-\cosh\mu_1)(\cosh k-\cosh\mu_2)\times \\ C^*(k)\mathrm{e}^{vy}\cosh mk\cos(\gamma x\sinh k)\mathrm{d}k \tag{77}$$

and

$$\chi_m^{\mathrm{II}a}=K_{m+2}(\gamma r)\sin(m+2)\theta+ \\ 2\frac{k_1+k_2}{\gamma}(K_{m+1}(\gamma r)\sin(m+1)\theta+K_{m-1}(\gamma r)\sin(m-1)\theta)+ \\ 2\left(1+2\frac{k_1k_2}{\gamma}\right)K_m(\gamma r)\sin m\theta+K_{m-2}(\gamma r)\sin(m-2)\theta+ \\ (-1)^m\int_0^\infty 4(\cosh k-\cosh\mu_1)(\cosh k-\cosh\mu_2)\times \\ C^*(k)\mathrm{e}^{vy}\sinh mk\sin(\gamma x\sinh k)\mathrm{d}k \tag{78}$$

3.2 Singularities in the upper layer

To develop multipoles singular at $y = f > 0$ and polar co-ordinates are again defined via (9). The solutions of Helmholtz equation singular at $y = f > 0$ are $K_n(\gamma r)\cos n\theta$ and $K_n(\gamma r)\sin n\theta$, $n \ge 1$. Let φ_n^s and φ_n^a denote the symmetric and antisymmetric multipoles satisfying (54) and (55) for upper and lower fluid except at (0, f) with boundary conditions (56) and (6) to (8) and represent an outgoing waves at infinity. Near the point (0, f), the behaviours of $\varphi_n^{\mathrm{I}s,a}$ are given by

$$\varphi_n^{\mathrm{I}s}\to K_n(\gamma r)\cos n\theta \quad \text{as} \quad r\to 0 \tag{79}$$

$$\varphi_n^{\mathrm{I}a}\to K_n(\gamma r)\sin n\theta \quad \text{as} \quad r\to 0 \tag{80}$$

Also they represent outgoing waves as $|x|\to\infty$.

The mutipoles are constructed as (Linton and Cadby, 2002)

$$\varphi_n^{\mathrm{I}s}=K_n(\gamma r)\cos n\theta+ \\ \int_0^\infty \cosh nk\cos(\gamma x\sinh k)\left(A_n^{(0)}(k)\mathrm{e}^{vy}+B_n^{(0)}(k)\mathrm{e}^{-vy}\right)\mathrm{d}k \tag{81}$$

$$\varphi_n^{\mathrm{II}s}=\int_0^\infty \cosh nk\cos(\gamma x\sinh k)C_n^{(0)}(k)\mathrm{e}^{vy}\mathrm{d}k \tag{82}$$

$$\varphi_n^{\mathrm{I}a}=K_n(\gamma r)\sin n\theta+ \\ \int_0^\infty \sinh nk\sin(\gamma x\sinh k)\left(A_n^{(1)}(k)\mathrm{e}^{vy}+B_n^{(1)}(k)\mathrm{e}^{-vy}\right)\mathrm{d}k \tag{83}$$

$$\varphi_n^{\mathrm{II}a}=\int_0^\infty \sinh nk\sin(\gamma x\sinh k)C_n^{(1)}(k)\mathrm{e}^{vy}\mathrm{d}k \tag{84}$$

where

$$A_n^{(j)}(k)=\frac{1}{2}\left[\left\{v\left(\sum_{m=0}^{m_0}(-1)^m c_m v^{2m}\right)+K\right\}\times\right. \\ \mathrm{e}^{-vh}\left\{(-1)^{n+j+1}((1-\rho)v-(1+\rho)K)\mathrm{e}^{vf}-\right. \\ \left.\left.(1-\rho)(v-K)\mathrm{e}^{-vf}\right\}\right]H(v) \tag{85}$$

$$B_n^{(j)}(k)=\frac{1}{2}\left[(-1)^{n+j+1}\left\{v\left(\sum_{m=0}^{m_0}(-1)^m c_m v^{2m}\right)+K\right\}\times\right. \\ ((1-\rho)(v-K)\mathrm{e}^{v(f-h)}-(1-\rho)(v-K)\times \\ \left.\left\{v\left(\sum_{m=0}^{m_0}(-1)^m c_m v^{2m}\right)-K\right\}\mathrm{e}^{-v(f-h)}\right]/H(v) \tag{86}$$

$$C_n^{(j)}(k)=-\rho K\left[(-1)^{n+j+1}\left\{v\left(\sum_{m=0}^{m_0}(-1)^m c_m v^{2m}\right)+K\right\}\mathrm{e}^{v(f-h)}-\right. \\ \left.\left\{v\left(\sum_{m=0}^{m_0}(-1)^m c_m v^{2m}\right)-K\right\}\mathrm{e}^{-v(f-h)}\right]/H(v),\ j=0,1 \tag{87}$$

where the contour is indented below the poles $k=\mu_1$ and $k=\mu_2$ in the complex k-plane.

The far-field forms of the multipoles, in the upper fluid, is given by

$$\varphi_n^{\mathrm{I}a}\sim a_n^{(4)}, \qquad \varphi_n^{\mathrm{I}a}\sim b_n^{(4)}$$

where

$$a_n^{(4)}=\mathrm{i}\pi[(A_n^{(0)}(k_1)\mathrm{e}^{k_1y}+B_n^{(0)}(k_1)\mathrm{e}^{-k_1y})\cosh n\mu_1\mathrm{e}^{\pm\mathrm{i}\beta_1x}+ \\ (A_n^{(0)}(k_2)\mathrm{e}^{k_2y}+B_n^{(0)}(k_2)\mathrm{e}^{-k_2y})\cosh n\mu_2\mathrm{e}^{\pm\mathrm{i}\beta_2x}] \tag{88}$$

$$b_n^{(4)}=\pi[(A_n^{(1)}(k_1)\mathrm{e}^{k_1y}+B_n^{(1)}(k_1)\mathrm{e}^{-k_1y})\sinh n\mu_1\mathrm{e}^{\pm\mathrm{i}\beta_1x}+ \\ (A_n^{(1)}(k_2)\mathrm{e}^{k_2y}+B_n^{(1)}(k_2)\mathrm{e}^{-k_2y})\sinh n\mu_2\mathrm{e}^{\pm\mathrm{i}\beta_2x}] \tag{89}$$

as $x\to\pm\infty$. Here $A_n^{(j)}(k_1)$, $A_n^{(j)}(k_2)$, and $B_n^{(j)}(k_1)$, $B_n^{(j)}(k_2)$ are the residues of $A_n^{(j)}(k)$ and $B_n^{(j)}(k)$ respectively at $k=\mu_1$ and $k=\mu_2$, given in (42) and (43).

Using (88) and (89), we find

$$a_{n+2}^{(4)}+2\frac{k_1+k_2}{\gamma}\left(a_{n+1}^{(4)}+a_{n-1}^{(4)}\right)+2\left(1+2\frac{k_1k_2}{\gamma}\right)a_n^{(4)}+a_{n-2}^{(4)}- \\ 8\mathrm{i}\pi\frac{k_1+k_2}{\gamma^2}\left((M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})\cosh n\mu_1\mathrm{e}^{\pm\mathrm{i}\beta_1x}+\right. \\ \left.(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})\cosh n\mu_2\mathrm{e}^{\pm\mathrm{i}\beta_2x}\right)=0 \tag{90}$$

and

$$b_{n+2}^{(4)}+2\frac{k_1+k_2}{\gamma}\left(b_{n+1}^{(4)}+b_{n-1}^{(4)}\right)+2\left(1+2\frac{k_1k_2}{\gamma}\right)b_n^{(4)}+b_{n-2}^{(4)}- \\ 8\pi\frac{k_1+k_2}{\gamma^2}\left((M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})\cosh n\mu_1\mathrm{e}^{\pm\mathrm{i}\beta_1x}+\right. \\ \left.(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})\sinh n\mu_2\mathrm{e}^{\pm\mathrm{i}\beta_2x}\right)=0 \tag{91}$$

Now using the representations (90) and (91) it can be shown that

$$\varphi_{n+2}^{\mathrm{ls}}+2\frac{k_1+k_2}{\gamma}\left(\varphi_{n+1}^{\mathrm{ls}}+\varphi_{n-1}^{\mathrm{ls}}\right)+2\left(1+2\frac{k_1k_2}{\gamma}\right)\varphi_n^{\mathrm{ls}}+\varphi_{n-2}^{\mathrm{ls}}-$$
$$8\mathrm{i}\pi\frac{k_1+k_2}{\gamma^2}\left((M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})\cosh n\mu_1\mathrm{e}^{\pm\mathrm{i}\beta_1x}+\right.$$
$$\left.(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})\cosh n\mu_2\mathrm{e}^{\pm\mathrm{i}\beta_2x}\right)=$$
$$K_{n+2}(\gamma r)\cos(n+2)\theta+$$
$$2\frac{k_1+k_2}{\gamma}(K_{n+1}(\gamma r)\cos(n+1)\theta+K_{n-1}(\gamma r)\cos(n-1)\theta)+$$
$$2\left(1+2\frac{k_1k_2}{\gamma}\right)K_n(\gamma r)\cos n\theta+K_{n-2}(\gamma r)\cos(n-2)\theta+ \tag{92}$$
$$(-1)^{n+1}\int_0^\infty 4(\cosh k-\cosh\mu_1)(\cosh k-\cosh\mu_2)\times$$
$$(M_1(k)\mathrm{e}^{vy}+N_1(k)\mathrm{e}^{-vy})\cosh nk\cos(\gamma x\sinh k)\mathrm{d}k+$$
$$\int_0^\infty 4(\cosh k+\cosh\mu_1)(\cosh k+\cosh\mu_2)\times$$
$$(M_2(k)\mathrm{e}^{vy}+N_2(k)\mathrm{e}^{-vy})\cosh nk\cos(\gamma x\sinh k)\mathrm{d}k-$$
$$8\mathrm{i}\pi\frac{k_1+k_2}{\gamma^2}\left((M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})\cosh n\mu_1\mathrm{e}^{\pm\mathrm{i}\beta_1x}+\right.$$
$$\left.(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})\cosh n\mu_2\mathrm{e}^{\pm\mathrm{i}\beta_2x}\right)$$

and

$$\varphi_{n+2}^{\mathrm{la}}+2\frac{k_1+k_2}{\gamma}\left(\varphi_{n+1}^{\mathrm{la}}+\varphi_{n-1}^{\mathrm{la}}\right)+2\left(1+2\frac{k_1k_2}{\gamma}\right)\varphi_n^{\mathrm{la}}+\varphi_{n-2}^{\mathrm{la}}-$$
$$8\pi\frac{k_1+k_2}{\gamma^2}\left((M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})\sinh n\mu_1\mathrm{e}^{\pm\mathrm{i}\beta_1x}+\right.$$
$$\left.(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})\sinh n\mu_2\mathrm{e}^{\pm\mathrm{i}\beta_2x}\right)=$$
$$K_{n+2}(\gamma r)\sin(n+2)\theta+$$
$$2\frac{k_1+k_2}{\gamma}(K_{n+1}(\gamma r)\sin(n+1)\theta+K_{n-1}(\gamma r)\sin(n-1)\theta)+$$
$$2\left(1+2\frac{k_1k_2}{\gamma}\right)K_n(\gamma r)\sin n\theta+K_{n-2}(\gamma r)\sin(n-2)\theta+ \tag{93}$$
$$(-1)^{n+2}\int_0^\infty 4(\cosh k-\cosh\mu_1)(\cosh k-\cosh\mu_2)\times$$
$$(M_1(k)\mathrm{e}^{vy}+N_1(k)\mathrm{e}^{-vy})\sinh nk\sin(\gamma x\sinh k)\mathrm{d}k+$$
$$\int_0^\infty 4(\cosh k+\cosh\mu_1)(\cosh k+\cosh\mu_2)\times$$
$$(M_2(k)\mathrm{e}^{vy}+N_2(k)\mathrm{e}^{-vy})\sinh nk\sin(\gamma x\sinh k)\mathrm{d}k-$$
$$8\pi\frac{k_1+k_2}{\gamma^2}\left((M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})\sinh n\mu_1\mathrm{e}^{\pm\mathrm{i}\beta_1x}+\right.$$
$$\left.(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})\sinh n\mu_2\mathrm{e}^{\pm\mathrm{i}\beta_2x}\right)$$

The last contour integrals in (92) and (93) can be written in the form

$$\int_0^\infty 4(\cosh k+\cosh\mu_1)(\cosh k+\cosh\mu_2)\times$$
$$(M_2(k)\mathrm{e}^{vy}+N_2(k)\mathrm{e}^{-vy})\cosh nk\cos(\gamma r\sin hk)\mathrm{d}k+$$
$$8\mathrm{i}\pi\frac{k_1+k_2}{\gamma^2}\left((M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})\cosh n\mu_1\mathrm{e}^{\pm\mathrm{i}\beta_1x}+\right. \tag{94}$$
$$\left.(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})\cosh n\mu_2\mathrm{e}^{\pm\mathrm{i}\beta_2x}\right)$$

$$\int_0^\infty 4(\cosh k+\cosh\mu_1)(\cosh k+\cosh\mu_2)\times$$
$$(M_2(k)\mathrm{e}^{vy}+N_2(k)\mathrm{e}^{-vy})\sinh nk\sin(\gamma r\sin hk)\mathrm{d}k+$$
$$8\pi\frac{k_1+k_2}{\gamma^2}\left((M_2(k_1)\mathrm{e}^{k_1y}+N_2(k_1)\mathrm{e}^{-k_1y})\sinh n\mu_1\mathrm{e}^{\pm\mathrm{i}\beta_1x}+\right. \tag{95}$$
$$\left.(M_2(k_2)\mathrm{e}^{k_2y}+N_2(k_2)\mathrm{e}^{-k_2y})\sinh n\mu_2\mathrm{e}^{\pm\mathrm{i}\beta_2x}\right)$$

the integrals being in the sense of Cauchy principal value.

After substituting (94), (95) in (92), (93) respectively and letting $f\to h$ in (92) and (93) we obtain the symmetric and antisymmetric wave-free potentials with singularity in the upper surface of the two-layer and are given by

$$\chi_m^{\mathrm{ls}}=K_{m+2}(\gamma r)\cos(m+2)\theta+$$
$$2\frac{k_1+k_2}{\gamma}(K_{m+1}(\gamma r)\cos(m+1)\theta+K_{m-1}(\gamma r)\cos(m-1)\theta)+$$
$$2\left(1+2\frac{k_1k_2}{\gamma}\right)K_m(\gamma r)\cos m\theta+K_{m-2}(\gamma r)\cos(m-2)\theta+$$
$$(-1)^m\int_0^\infty 4(\cosh k-\cosh\mu_1)(\cosh k-\cosh\mu_2)\times \tag{96}$$
$$(M_1^*(k)\mathrm{e}^{vy}+N_1^*(k)\mathrm{e}^{-vy})\cosh mk\cos(\gamma x\sinh k)\mathrm{d}k+$$
$$\int_0^\infty 4(\cosh k+\cosh\mu_1)(\cosh k+\cosh\mu_2)\times$$
$$(M_2^*(k)\mathrm{e}^{vy}+N_2^*(k)\mathrm{e}^{-vy})\cosh mk\cos(\gamma x\sinh k)\mathrm{d}k$$

and

$$\chi_m^{\mathrm{la}}=K_{m+2}(\gamma r)\sin(m+2)\theta+$$
$$2\frac{k_1+k_2}{\gamma}(K_{m+1}(\gamma r)\sin(m+1)\theta+K_{m-1}(\gamma r)\sin(m-1)\theta)+$$
$$2\left(1+2\frac{k_1k_2}{\gamma}\right)K_m(\gamma r)\sin m\theta+K_{m-2}(\gamma r)\sin(m-2)\theta+$$
$$(-1)^m\int_0^\infty 4(\cosh k-\cosh\mu_1)(\cosh k-\cosh\mu_2)\times \tag{97}$$
$$(M_1^*(k)\mathrm{e}^{vy}+N_1^*(k)\mathrm{e}^{-vy})\sinh mk\sin(\gamma x\sinh k)\mathrm{d}k+$$
$$\int_0^\infty 4(\cosh k+\cosh\mu_1)(\cosh k+\cosh\mu_2)\times$$
$$(M_2^*(k)\mathrm{e}^{vy}+N_2^*(k)\mathrm{e}^{-vy})\sinh mk\sin(\gamma x\sinh k)\mathrm{d}k$$

the last integrals in (96) and (97) being in the sense of Cauchy principal value.

In particular, choose c_0=1, c_i=0, i=1, 2,…, m_0, the BVP becomes the BVP for two-layer fluid with free surface (Linton and Cadby, 2002) and the multipoles exactly coincide with those for the case of two-layer fluid with free surface (Linton and Cadby, 2002) and the wave-free potentials become the wave-free potentials for two-layer fluid with free surface. Similarly, if choose $c_0=1-\varepsilon K$, c_1=0, $c_2=D$, c_i=0, i=3, 4,…, m_0, then the BVP becomes the BVP for two layer fluid with ice-cover boundary condition (3) (Das and Mandal, 2007) and obtain the corresponding multipoles (Das and Mandal, 2007) and wave-free potentials. If we let c_0=1, c_i=0, i=1, 2,…, m_0 and $\rho\to 0$ in this problem then it can be shown that the multipoles and wave-free potential functions go over to the single layer multipoles evaluated by Thorne (1953) and wave-free potential evaluated by Das and Mandal (2010a). Thus by letting $\rho\to 0$ in the above analysis we recover the results for the single layer fluid.

4 Three dimensional multipoles and wave-free potentials

Here the velocity potential

$$\Phi(x,y,z,t)=\mathrm{Re}\{\varphi(x,y,z)\mathrm{e}^{-\mathrm{i}\omega t}\}$$

describing the fluid motion exists where $\varphi(x,y,z)$ is a complex valued function and ω is the angular frequency. Let the potential in the upper layer be φ_{I}^{m} and that in the lower layer be $\varphi_{\mathrm{II}}^{m}$ (m=0, 1, the potential functions for the heave and sway problems being denoted by φ^0 and φ^1, respectively). The potential functions satisfy the Laplace's Eqs. (1), (2) with

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

and equations (6) and (7) represent the linearized boundary conditions at the interface y=0, while the free-surface condition with higher-order derivatives at y=h is

$$\mathcal{N}\varphi_y^1-K\varphi^1=0 \qquad \text{on} \quad y=h \tag{98}$$

where the differential operator

$$\mathcal{N}=\sum_{m=0}^{m_0}c_m\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)^m \tag{99}$$

where c_m (m=0, 1,…, m_0) are known constants. Keeping in mind various physical problems involving fluid structure interaction, only the even order partial derivatives in x are considered in the differential operator $\mathcal{N}$ and (8) is the bottom condition of the lower layer.

4.1 Singularities in the lower layer

The velocity potential singular at (0, f, 0) describing the motion in the lower fluid, then multipole singular potential functions are solutions of the Laplace's equation which are singular at r=0, satisfy the boundary conditions (6) to (8) and (98). These can be constructed using the method given by Thorne (1953). A solution of Laplace's equation in the spherical polar co-ordinate system (r, θ, α) and singular at r=0 is $r^{-n-1}P_n^m(\cos\theta)\cos m\alpha,\ n\geq m\geq 0$, where P_n^m are associated Legendre functions. This has the integral representation, valid for y>f (Thorne, 1953)

$$\begin{aligned}&\frac{P_n^m(\cos\theta)}{r^{n+1}}\cos m\alpha=\\&\frac{\cos m\alpha}{(n-m)!}\int_0^\infty k^n\mathrm{e}^{-k(y-f)}J_m(kR)\mathrm{d}k\end{aligned} \tag{100}$$

where J_m are Bessel functions and $R=(x^2+z^2)^{1/2}$. Let the multipole potentials $\varphi_{\mathrm{I}n}^{m}\cos m\alpha$ and $\varphi_{\mathrm{II}n}^{m}\cos m\alpha$ (in the notation of Cadby and Linton (2000), m=0, 1) be the singular solutions of the Laplace's equation and satisfy the free surface boundary condition with higher order derivatives (98), the interface conditions (6) and (7) and behave as outgoing waves as $R\to\infty$. Then $\varphi_{\mathrm{I}n}^{m}$ and $\varphi_{\mathrm{II}n}^{m}$ are obtained as

$$\begin{aligned}&\varphi_{\mathrm{I}n}^{m}=\frac{a^{n+1}}{(n-m)!}\int_0^\infty k^n\left(A(k)\mathrm{e}^{ky}+B(k)\mathrm{e}^{-ky}\right)J_m(kR)\mathrm{d}k\\&\qquad 0<y<h\end{aligned} \tag{101}$$

$$\begin{aligned}&\varphi_{\mathrm{II}n}^{m}=\left(\frac{a}{r}\right)^{n+1}P_n^m(\cos\theta)+\\&\frac{a^{n+1}}{(n-m)!}\int_0^\infty k^nC(k)\mathrm{e}^{ky}J_m(kR)\mathrm{d}k \quad y<0\end{aligned} \tag{102}$$

where

$$A(k)=K\left\{k\left(\sum_0^{m_0}(-1)^mc_mk^{2m}\right)+K\right\}\mathrm{e}^{k(f-h)}/H(k) \tag{103}$$

$$B(k)=K\left\{k\left(\sum_0^{m_0}(-1)^mc_mk^{2m}\right)-K\right\}\mathrm{e}^{k(f+h)}/H(k) \tag{104}$$

$$\begin{aligned}&C(k)=\left[K\rho\left\{k\left(\sum_0^{m_0}(-1)^mc_mk^{2m}\right)\cosh kh-K\sinh kh\right\}-\right.\\&\left.\{(1-\rho)k+K\}\left\{k\left(\sum_0^{m_0}(-1)^mc_mk^{2m}\right)\sinh kh-K\cosh kh\right\}\right]\times\\&\mathrm{e}^{kf}/H(k)\end{aligned} \tag{105}$$

The path of integration in the integrals in (101), (102) is chosen to be indented below the poles at k=k_1 and k=k_2. It can be shown that as $R\to\infty$, only the contributions to the integrals from these indentations, to the potential functions φ, prevail and behave as waves having outgoing nature.

The far-field forms of the multipoles, in the lower layer, is given by

$$\varphi_{\mathrm{II}n}^{m}\sim a_n^m\left(\frac{2\pi}{R}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}$$

as $R\to\infty$, where

$$\begin{aligned}&a_n^m=\frac{(-\mathrm{i})^{m+1}a^{n+1}}{(n-m)!}\times\\&(k_1^{n-\frac{1}{2}}C_{k_1}\mathrm{e}^{\mathrm{i}k_1R+k_1y}+k_2^{n-\frac{1}{2}}C_{k_2}\mathrm{e}^{\mathrm{i}k_2R+k_2y})\end{aligned} \tag{106}$$

C_{k_1} and C_{k_2} being the residues of $C(k)$ at k=k_1 and k=k_2 respectively, which are given by

$$\begin{aligned}&C_{k_j}=\left[K\rho\left\{k_j\left(\sum_0^{m_0}(-1)^mc_mk_j^{2m}\right)\cosh k_jh-K\sinh k_jh\right\}-\right.\\&\left.\{(1-\rho)k_j+K\}\left\{k_j\left(\sum_0^{m_0}(-1)^mc_mk_j^{2m}\right)\sinh k_jh-K\cosh k_jh\right\}\right]\times\\&\mathrm{e}^{k_jf}/H'(k_j) \quad j=1,\ 2\end{aligned} \tag{107}$$

Using (106), we find

$$\begin{aligned}&a_{n+2}^m-\frac{k_1+k_2}{n-m+2}a_{n+1}^m+\\&\frac{k_1k_2}{(n-m+2)(n-m+1)}a_n^m=0\end{aligned} \tag{108}$$

Now using the representations (108), it can be shown that

$$\varphi^m_{\mathrm{II}(n+2)} - \frac{k_1+k_2}{n-m+2}\varphi^m_{\mathrm{II}(n+1)} + \frac{k_1k_2}{(n-m+1)(n-m+2)}\varphi^m_{\mathrm{II}n} = \left(\frac{a}{r}\right)^{n+3} P^m_{(n+2)}(\cos\theta) - \frac{k_1+k_2}{n-m+2}\left(\frac{a}{r}\right)^{n+2} P^m_{(n+1)}(\cos\theta) + \frac{k_1k_2}{(n-m+2)(n-m+1)}\left(\frac{a}{r}\right)^{n+1} P^m_n(\cos\theta) + \frac{a^{n+3}}{(n-m+2)!}\times \int_0^\infty (k-k_1)(k-k_2)k^n C(k)\mathrm{e}^{ky} J_m(kR)\mathrm{d}k \quad (109)$$

This is the wave-free potential having singularity at (0, *f*, 0). Letting $f\to 0$ in (109) it is obtained the wave-free potentials having singularity near the interface between two-layer and is given by

$$\chi^m_{\mathrm{II}n} = \left(\frac{a}{r}\right)^{n+3} P^m_{(n+2)}(\cos\theta) - \frac{k_1+k_2}{n-m+2}\left(\frac{a}{r}\right)^{n+2} P^m_{(n+1)}(\cos\theta) + \frac{k_1k_2}{(n-m+2)(n-m+1)}\left(\frac{a}{r}\right)^{n+1} P^m_n(\cos\theta) + \frac{a^{n+3}}{(n-m+2)!}\times \int_0^\infty (k-k_1)(k-k_2)k^n C^*(k)\mathrm{e}^{ky} J_m(kR)\mathrm{d}k \quad (110)$$

4.2 Singularities in the upper layer

To develop multipoles singular at $y = f > 0$. The solution of Laplace's equation singular at $y = f > 0$ is $r^{-n-1}P^m_n(\cos\theta)\cos m\alpha$. Let $\varphi^{m,s}_n$ is denote the multipoles satisfying (1), (2) except at (0, *f*, 0) with boundary conditions (6), (7) and (98). The mutipoles are constructed as (Cadby and Linton, 2000)

$$\varphi^m_{\mathrm{I}n} = \left(\frac{a}{r}\right)^{n+1} P^m_n(\cos\theta) + \frac{(-1)^{m+n}a^{n+1}}{(n-m)!}\times \int_0^\infty k^n\left(A_1(k)\mathrm{e}^{ky} + B_1(k)\mathrm{e}^{-ky}\right)J_m(kR)\mathrm{d}k \quad (111)$$

$$\varphi^m_{\mathrm{II}n} = \frac{(-1)^{m+n}a^{n+1}}{(n-m)!}\int_0^\infty k^n C_1(k)\mathrm{e}^{ky}J_m(kR)\mathrm{d}k \quad (112)$$

where

$$A_1(k) = \frac{1}{2}\left[\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right) + K\right\}\mathrm{e}^{-kf}\times \left\{(-1)^{n+m+1}((1-\rho)k-(1-\rho)K)\mathrm{e}^{kf} - (1-\rho)(k-K)\mathrm{e}^{-kf}\right\}\right] / H(k) \quad (113)$$

$$B_1(k) = \frac{1}{2}\left[(-1)^{n+m+1}\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right) + K\right\}\times (1-\rho)(k-K)\mathrm{e}^{k(f-h)} - (1-\rho)(k-K)\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right) - K\right\}\mathrm{e}^{-k(f-h)}\right] / H(k) \quad (114)$$

$$C_1(k) = -\rho K\left[(-1)^{n+m+1}\left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right) + K\right\}\mathrm{e}^{k(f-h)} - \left\{k\left(\sum_0^{m_0}(-1)^m c_m k^{2m}\right) - K\right\}\mathrm{e}^{-k(f-h)}\right] / H(k),\ m = 0,1 \quad (115)$$

The path of the integration in the integrals in (111) and (112) is indented below the poles at k=k_1 and k=k_2 on the real k-axis.

The far-field forms of the multipoles, in the upper fluid, is given by

$$\varphi^m_{\mathrm{I}n} \sim b^m_n$$

where

$$b^m_n = \frac{(-1)^n \mathrm{i}^{m+1}a^{n+1}}{(n-m)!}\left(\frac{2\pi}{R}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}[k_1^{n-1/2}(A_1(k_1)\mathrm{e}^{k_1 y} + B_1(k_1)\mathrm{e}^{-k_1 y})\mathrm{e}^{\mathrm{i}k_1 R} + k_2^{n-1/2}(A_1(k_2)\mathrm{e}^{k_2 y} + B_1(k_2)\mathrm{e}^{-k_2 y})\mathrm{e}^{\mathrm{i}k_2 R}] \quad (116)$$

As $R\to\infty$. Here $A_1(k_1)$, $A_1(k_2)$ and $B_1(k_1)$, $B_1(k_2)$ are the residues of $A_1(k)$ and $B_1(k)$ respectively at k=k_1 and k=k_2, given by

$$A_1(k_s) = (-1)^{n+m+1}M_1(k_s) + M_2(k_s) \quad s = 1,2 \quad (117)$$

$$B_1(k_s) = (-1)^{n+m+1}N_1(k_s) + N_2(k_s) \quad s = 1,2 \quad (118)$$

Using (116), it is found that

$$b^m_{n+2} + \frac{a(k_1+k_2)}{n-m+2}b^m_{n+1} + \frac{a^2k_1k_2}{(n-m+1)(n-m+2)}b^m_n + \frac{(-1)^n\mathrm{i}^{m+1}a^{n+3}}{(n-m+2)!}(k_1+k_2)\left(\frac{2\pi}{R}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\times \{(M_1(k_1)\mathrm{e}^{k_1 y} + N_1(k_1)\mathrm{e}^{-k_1 y})k_1^{n+1}\mathrm{e}^{\mathrm{i}k_1 R} + (M_1(k_2)\mathrm{e}^{k_2 y} + N_1(k_2)\mathrm{e}^{-k_2 y})k_1^{n+1}\mathrm{e}^{\mathrm{i}k_2 R}\} = 0 \quad (119)$$

Now using the representation (119) it can be shown that

$$\varphi^m_{\mathrm{I}(n+2)} + \frac{a(k_1+k_2)}{n-m+2}\varphi^m_{\mathrm{I}(n+1)} + \frac{a^2k_1k_2}{(n-m+1)(n-m+2)}\varphi^m_{\mathrm{I}n} + \frac{(-1)^n\mathrm{i}^{m+1}a^{n+3}}{(n-m+2)!}(k_1+k_2)\left(\frac{2\pi}{R}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\times \{(M_2(k_1)\mathrm{e}^{k_1 y} + N_2(k_1)\mathrm{e}^{-k_1 y})k_1^{n+1}\mathrm{e}^{\mathrm{i}k_1 R} + (M_1(k_2)\mathrm{e}^{k_2 y} + N_1(k_2)\mathrm{e}^{-k_2 y})k_2^{n+1}\mathrm{e}^{\mathrm{i}k_2 R}\} = \left(\frac{a}{r}\right)^{n+3}P^m_{(n+2)}(\cos\theta) + \frac{a(k_1+k_2)}{n-m+2}\left(\frac{a}{r}\right)^{n+2}P^m_{(n+1)}(\cos\theta) + \frac{a^2k_1k_2}{(n-m+1)(n-m+2)}\left(\frac{a}{r}\right)^{n+1}P^m_n(\cos\theta) + \frac{(-1)^{m+n}a^{n+3}}{(n-m+2)!}\int_0^\infty k^n(k-k_1)(k-k_2)\times [(M_2(k)\mathrm{e}^{ky} + N_2(k)\mathrm{e}^{-ky})]J_m(kR)\mathrm{d}k - \frac{a^{n+3}}{(n-m+2)!}\int_0^\infty k^{n+1}(k+k_1)(k+k_2)\times [(M_1(k)\mathrm{e}^{ky} + N_1(k)\mathrm{e}^{-ky})]J_m(kR)\mathrm{d}k + \frac{(-1)^n\mathrm{i}^{m+1}a^{n+3}}{(n-m+2)!}(k_1+k_2)\left(\frac{2\pi}{R}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\times \{(M_1(k_1)\mathrm{e}^{k_1 y} + N_1(k_1)\mathrm{e}^{-k_1 y})k_1^{n+1}\mathrm{e}^{\mathrm{i}k_1 R} + (M_1(k_2)\mathrm{e}^{k_2 y} + N_1(k_2)\mathrm{e}^{-k_2 y})k_1^{n+1}\mathrm{e}^{\mathrm{i}k_2 R}\} \quad (120)$$

The last contour integrals in (120) can be written in the form:

$$\frac{a^{n+3}}{(n-m+2)!}\int_0^\infty k^{n+1}(k+k_1)(k+k_2)\times$$
$$[(M_1(k)\mathrm{e}^{ky}+N_1(k)\mathrm{e}^{-ky})]J_m(kR)\mathrm{d}k+$$
$$\frac{(-1)^n\mathrm{i}^{m+1}a^{n+3}}{(n-m+2)!}(k_1+k_2)\left(\frac{2\pi}{R}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\times \quad (121)$$
$$\{(M_1(k_1)\mathrm{e}^{k_1y}+N_1(k_1)\mathrm{e}^{-k_1y})k_1^{n+1}\mathrm{e}^{\mathrm{i}k_1R}+$$
$$(M_1(k_2)\mathrm{e}^{k_2y}+N_1(k_2)\mathrm{e}^{-k_2y})k_1^{n+1}\mathrm{e}^{\mathrm{i}k_2R}\}=0$$

the integral being in the sense of Cauchy principal value.

Thus (120) reduces to

$$\varphi_{1(n+2)}^m+\frac{a(k_1+k_2)}{n-m+2}\varphi_{1(n+1)}^m+\frac{a^2k_1k_2}{(n-m+1)(n-m+2)}\varphi_{1n}^m+$$
$$\frac{(-1)^n\mathrm{i}^{m+1}a^{n+3}}{(n-m+2)!}(k_1+k_2)\left(\frac{2\pi}{R}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\times$$
$$\{(M_1(k_1)\mathrm{e}^{k_1y}+N_1(k_1)\mathrm{e}^{-k_1y})k_1^{n+1}\mathrm{e}^{\mathrm{i}k_1R}+$$
$$(M_1(k_2)\mathrm{e}^{k_2y}+N_1(k_2)\mathrm{e}^{-k_2y})k_2^{n+1}\mathrm{e}^{\mathrm{i}k_2R}\}=$$
$$\left(\frac{a}{r}\right)^{n+3}P_{(n+2)}^m(\cos\theta)+\frac{a(k_1+k_2)}{n-m+2}\left(\frac{a}{r}\right)^{n+2}P_{(n+1)}^m(\cos\theta)+$$
$$\frac{a^2k_1k_2}{(n-m+2)(n-m+1)}\left(\frac{a}{r}\right)^{n+1}P_n^m(\cos\theta)+ \quad (122)$$
$$\frac{(-1)^{m+n}a^{n+3}}{(n-m+2)!}\int_0^\infty k^n(k-k_1)(k-k_2)\times$$
$$[(M_2(k)\mathrm{e}^{ky}+N_2(k)\mathrm{e}^{-ky})]J_m(kR)\mathrm{d}k-$$
$$\frac{a^{n+3}}{(n-m+2)!}\int_0^\infty k^{n+1}(k+k_1)(k+k_2)\times$$
$$[(M_1(k)\mathrm{e}^{ky}+N_1(k)\mathrm{e}^{-ky})]J_m(kR)\mathrm{d}k$$

This is the wave-free potential having singularity at (0, f, 0). Letting $f\to h$ in (122) it is obtained the wave-free potentials with singularity in the upper surface of the two-layer and are given by

$$\chi_{1n}^m=\left(\frac{a}{r}\right)^{n+3}P_{(n+2)}^m(\cos\theta)+$$
$$\frac{a(k_1+k_2)}{n-m+2}\left(\frac{a}{r}\right)^{n+2}P_{(n+1)}^m(\cos\theta)+$$
$$\frac{a^2k_1k_2}{(n-m+2)(n-m+1)}\left(\frac{a}{r}\right)^{n+1}P_n^m(\cos\theta)+$$
$$\frac{(-1)^{m+n}a^{n+3}}{(n-m+2)!}\cdot\int_0^\infty k^n(k-k_1)(k-k_2)\times \quad (123)$$
$$[(M_2^*(k)\mathrm{e}^{ky}+N_2^*(k)\mathrm{e}^{-ky})]J_m(kR)\mathrm{d}k-$$
$$\frac{a^{n+3}}{(n-m+2)!}\cdot\int_0^\infty k^{n+1}(k+k_1)(k+k_2)\times$$
$$[(M_1^*(k)\mathrm{e}^{ky}+N_1^*(k)\mathrm{e}^{-ky})]J_m(kR)\mathrm{d}k$$

In particular, choose c_0=1, c_i=0, i=1, 2,…, m_0, the BVP becomes the BVP for two-layer fluid with free surface (Cadby and Linton, 2000) and the multipoles exactly coincide with those for the case of two-layer fluid with free surface (Cadby and Linton, 2000) and the wave-free potentials become the wave-free potentials for two-layer fluid with free surface. Similarly, if choose $c_0=1-\varepsilon K$, c_1=0, $c_2=D$, c_i=0, i=3, 4,…, m_0, then the BVP becomes the BVP for two layer fluid with ice-cover boundary condition (3) (Das and Mandal, 2010b) and obtain the corresponding multipoles (Das and Mandal, 2010b) and wave-free potentials. If we let c_0=1, c_i=0, i=1, 2,…, m_0 and $\rho\to 0$ in this problem then it can be shown that the multipoles and wave-free potential functions go over to the single layer multipoles evaluated by Thorne (1953) and wave-free potential evaluated by Dhillon and Mandal (2013). Thus by letting $\rho\to 0$ in the above analysis we recover the results for the single layer fluid.

5 Conclusions

Wave-free potentials and multipoles in two-layer fluid with a free surface condition with higher order derivatives for non-oblique and oblique waves (two dimensions) and also three dimension are constructed in a symmetric manner. Appropriate modifications of the wave-free potentials can be made in the circumstances when the two-layer fluid are of uniformly finite depth for both the layers having a free surface conditions with higher order derivatives. In particular, these are obtained taking into account of the effect of the presence of surface tension at the free surface and also in the presence of an ice-cover modelled as a thin elastic plate. Also for limiting case, it can be shown that the multipoles and wave-free potential functions go over to the single layer multipoles and wave-free potential.

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