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Scattering of Surface Water Waves Involving Semi-infinite Floating Elastic Plates on Water of Finite Depth

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Abstract: Two problems of scattering of surface water waves involving a semi-infinite elastic plate and a pair of semi-infinite elastic plates, separated by a gap of finite width, floating horizontally on water of finite depth, are investigated in the present work for a two-dimensional time-harmonic case. Within the frame of linear water wave theory, the solutions of the two boundary value problems under consideration have been represented in the forms of eigenfunction expansions. Approximate values of the reflection and transmission coefficients are obtained by solving an over-determined system of linear algebraic equations in each problem. In both the problems, the method of least squares as well as the singular value decomposition have been employed and tables of numerical values of the reflection and transmission coefficients are presented for specific choices of the parameters for modelling the elastic plates. Our main aim is to check the energy balance relation in each problem which plays a very important role in the present approach of solutions of mixed boundary value problems involving Laplace equations. The main advantage of the present approach of solutions is that the results for the values of reflection and transmission coefficients obtained by using both the methods are found to satisfy the energy-balance relations associated with the respective scattering problems under consideration. The absolute values of the reflection and transmission coefficients are presented graphically against different values of the wave numbers.

Keywords: surface water waves; floating elastic plates; over-determined systems; least squares method; singular value decomposition method; scattering problem; reflection and transmission coefficients

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1 Introduction

Scattering of surface water waves by very large floating structures (VLFS) have been studied in recent decades. VLFSs can serve as various offshore constructions, such as floating airports, oil storage facilities, wind and solar power plants, etc. Accordingly, the elastic deformations are predominant over the rigid body motions in the response of the structure to water waves. Thus VLFSs are usually modelled as elastic plates. It is noted that the ice sheets in the polar region can also be considered as thin elastic plates. The interaction between water waves and elastic plates has been a subject of interest for both VLFS designers as well as polar marine researchers. Scattering of surface water waves by floating elastic plates, on the surface of water of finite depth, creates interesting mathematical problems drawing attention of various types for obtaining their useful solutions, see Newman (1965), Evans (1985), Blamforth and Craster (1999), Chakrabarti (2000), Sahoo *et al.* (2001).

Various approaches have been developed in the literature to deal with the interaction of water waves with elastic plates. It is well known that the method of eigenfunction expansions was often used in analyzing engineering problems. This method originates from the method of separation of variables, the pivotal step of which is to determine the unknown expansion coefficients. In the present paper, we have explained certain algebraic approaches (viz. the least squares and singular value decomposition methods) to determine the unknown expansion coefficients in such expansion procedures. Two special water waves scattering problems have been analyzed in the light of the algebraic approaches explained in the present work. It is emphasized that the energy balance relation for any water waves scattering problem is of great use in checking the correctness of the analytical as well as numerical results determining the reflection and transmission coefficients. Therefore, our main aim is to check the energy balance relations for the water waves scattering problems involving floating elastic plates by applying the present approach of solutions.

2 The algebraic approaches

All the problems of expansion are reduced to the over-determined system $\boldsymbol{Ax}=\boldsymbol{b}$ and the two basic approaches are as follows:

2.1 The least squares method

Consider an over-determined system of linear algebraic equations as given by

$$\boldsymbol{Ax}=\boldsymbol{b} \tag{1}$$

where $\boldsymbol{A}$ is an $m\times n\,(m>n)$ real or complex matrix, $\boldsymbol{x}\in R^n$ or C^n, and $\boldsymbol{b}\in R^m$ or C^m.

The algebraic least squares solution to the system denoted

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by $\boldsymbol{y} \in R^n$ or C^n, is the solution to the associated normal system:

$$\boldsymbol{A}^* \boldsymbol{A} \boldsymbol{y} = \boldsymbol{A}^* \boldsymbol{b} \tag{2}$$

where $\boldsymbol{A}^*$ denotes the conjugate transpose of $\boldsymbol{A}$. Further, if $\boldsymbol{A}$ has linearly independent columns, then there is a unique Least Squares solution (Miller, 1973), as given by

$$\boldsymbol{y} = \left(\boldsymbol{A}^* \boldsymbol{A}\right)^{-1} \boldsymbol{b} \tag{3}$$

2.2 Singular Value Decomposition method

We often resort to singular value decomposition (SVD) of the matrix $\boldsymbol{A}$ as a tool to solve the over-determined system of Eq. (1). It is well known (Strang, 1998) that SVD is the factorization of a matrix into a product of three matrices that reveal the structure of the original matrix.

Now the SVD of an $m \times n$ real or complex matrix $\boldsymbol{A}$ is a factorization of the form:

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^* \tag{4}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}^*$ (the conjugate transpose of $\boldsymbol{V}$) are real or complex unitary matrices of order $m \times m$ and $n \times n$ respectively, and $\boldsymbol{S}$ is an $m \times n$ diagonal matrix with non-negative real numbers on the diagonal entries. The columns of $\boldsymbol{U}$ and $\boldsymbol{V}$ are the eigenvectors of $\boldsymbol{A}\boldsymbol{A}^*$ and $\boldsymbol{A}^*\boldsymbol{A}$ respectively, and the diagonal entries of $\boldsymbol{S}$ are the square roots of the non-zero eigenvalues of $\boldsymbol{A}^*\boldsymbol{A}$ or $\boldsymbol{A}\boldsymbol{A}^*$, which are known as the singular values of $\boldsymbol{A}$.

Replacing $\boldsymbol{A}$ in the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ by using Eq. (4), we get

$$\boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^* \boldsymbol{x} = \boldsymbol{b} \tag{5}$$

$$\boldsymbol{S}\boldsymbol{V}^* \boldsymbol{x} = \boldsymbol{U}^* \boldsymbol{b} \tag{6}$$

Then setting

$$\boldsymbol{x} = \boldsymbol{V}\boldsymbol{y} \tag{7}$$

we get

$$\boldsymbol{S}\boldsymbol{y} = \boldsymbol{U}^* \boldsymbol{b} \tag{8}$$

and the solution of the system (1) can finally be determined by using the relations (7) and (8).

Having introduced the two methods for solving an over-determined system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, as described above, it must be noted that for the case of determinate system, we may use other known methods of solution involving the determination of the inverse matrix $\boldsymbol{A}^{-1}$, whenever it exists.

It is emphasized that in order to be able to utilize the algebraic approaches described above successfully, occurrence of ill-conditioned matrices must be avoided. For the problems chosen in the present work which will be taken up in the next sections we have avoided such problems by choosing appropriate points of discretization.

In the next two sections, we present the solutions of two special water waves scattering problems, namely, scattering of water waves by a semi-infinite elastic plate and a pair of semi-infinite elastic plates, separated by a gap of finite width, floating horizontally on water of finite depth. Moreover, we check the energy balance relation for both the scattering problems for checking the correctness of the analytical as well as numerical results determining the reflection and transmission coefficients.

3 Water wave scattering by a semi-infinite elastic plate

In this section, we consider the problem of scattering of water waves involving an ocean of finite depth having a flat rigid bottom surface, whereas the upper surface of the ocean is bounded above by a thin uniform semi-infinite elastic plate modelled as a thin elastic ice-cover (Ursell, 1947; Stoker, 1957; Weitz and Keller, 1950; Newman, 1965; Evans, 1985; Evans and Linton, 1994; Blamforth and Craster, 1999; Chakrabarti, 2000; Chung and Fox, 2002; Sahoo *et al.*, 2001). In this case, different boundary conditions on the upper surface are satisfied on different sides of the line of discontinuity, constituting the edge of a thin semi-infinite elastic plate floating on the surface. Time-harmonic waves of a particular frequency are assumed to propagate normally to the edge of the floating plate. Using the algebraic approaches involving the least squares and singular value decomposition methods, we have determined the reflection and transmission coefficients approximately.

We consider the irrotational motion of an inviscid incompressible fluid of relatively small amplitude under the action of gravity and is covered by a thin uniform semi-infinite elastic plate modelled as a thin ice sheet. The fluid is of infinite horizontal extent in x-direction while the depth is along y-direction which is considered vertically upward with $y = 0$ as the mean position of the top surface and $y = -h$ as the bottom surface. We assume that a floating plate occupies the region $y = 0, x > 0$, whereas the region $y = 0, x < 0$, is free to the upper atmosphere. We further assume that all motions are time harmonic with angular frequency ω. Under these assumptions, the velocity potential of the flow can be written as $\mathrm{Re}\left[\phi(x,y)\mathrm{e}^{-\mathrm{i}\omega t}\right]$, where Re stands for the real part, and the potential function $\phi(x,y)$ satisfies Laplace's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{in} \quad -\infty < x < \infty, \; -h < y < 0 \tag{9}$$

with the linearized boundary conditions on the top surface and bottom surface being given by:

$$\frac{\partial \phi}{\partial y} - K\phi = 0 \quad \text{on} \quad y = 0, \; -\infty < x < 0 \tag{10}$$

$$\left(D\frac{\partial^4}{\partial x^4} + 1 - \varepsilon K\right)\frac{\partial \phi}{\partial y} - K\phi = 0, \quad \text{on} \quad y = 0, 0 < x < \infty \tag{11}$$

$$\frac{\partial^3 \phi}{\partial x^2 \partial y} \to 0, \; \frac{\partial^4 \phi}{\partial x^3 \partial y} \to 0, \quad \text{as} \quad x \to 0^-, y \to 0^- \tag{12}$$

$$\phi(x,y) \sim \begin{cases} \left(e^{ik_0 x} + R e^{-ik_0 x}\right) \dfrac{\cosh k_0 (h+y)}{\cosh k_0 h}, \text{ as } x \to -\infty \\ T e^{ip_0 x} \dfrac{\cosh p_0 (h+y)}{\cosh p_0 h}, \text{ as } x \to \infty \end{cases} \tag{13}$$

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad y = -h, \ -\infty < x < \infty \tag{14}$$

where $K = \omega^2/g$; g the acceleration due to gravity, $D = (L/\rho g)$; L is the flexural rigidity of the elastic plate, $\varepsilon = (\rho_0/\rho) h_0$; ρ_0 is the density of the plate, ρ is the density of fluid, h_0 is the very small thickness of the elastic plate, R and T are the unknown complex constants which are related with the reflection and transmission coefficients respectively, k_0 and p_0 are positive real numbers satisfying, respectively, the dispersion relations:

$$K - u \tanh uh = 0 \tag{15}$$

and

$$\left(Dv^4 + 1 - \varepsilon K\right) v \tanh vh - K = 0 \tag{16}$$

Now it may be noted here that Eqs. (15) and (16), respectively, represent the dispersion relations for the open region $(x < 0, -h < y < 0)$ and the plate covered region $(x > 0, -h < y < 0)$. In equation (15), there are two non-zero real roots $\pm k_0$ that indicate the propagating modes and a countable infinity of purely imaginary roots $\pm iu_n (n = 1,2,...)$ that relate to a set of evanescent modes, where u_n's are real and positive satisfying the equation

$$K + u_n \tan u_n h = 0. \tag{17}$$

Similarly, in equation (16), there are two non-zero real roots $\pm p_0$ that indicate the propagating modes; two complex conjugate pair of roots $\pm(\alpha \pm i\beta)$ with $\alpha, \beta > 0$, corresponding to the damped propagating modes; and a countable infinity of purely imaginary roots $\pm iv_n (n = 1,2,...)$ that relate to a set of evanescent modes, where v_n's are real and positive satisfying the equation

$$\left(Dv_n^4 + 1 - \varepsilon K\right) v_n \tan v_n h + K = 0 \tag{18}$$

3.1 Method of solution

The entire fluid domain is divided into two regions: the open region $(x < 0, -h < y < 0)$ and plate covered region $(x > 0, -h < y < 0)$. The velocity potential in the fluid region is expressed as

$$\phi(x,y) \sim \begin{cases} \left(e^{ik_0 x} + R e^{-ik_0 x}\right) \psi_0(y) + \displaystyle\sum_{m=1}^{\infty} A_m e^{k_m x} \psi_m(y), \text{ for } x < 0 \\ T e^{ip_0 x} f_0(y) + \displaystyle\sum_{\substack{n=-2 \\ n \neq 0}}^{\infty} B_n e^{-p_n x} f_n(y), \text{ for } x > 0 \end{cases} \tag{19}$$

where

$$\psi_m(y) = \begin{cases} \dfrac{\cosh k_0 (h+y)}{\cosh k_0 h}, \text{ for } m = 0 \\ \dfrac{\cos k_m (h+y)}{\cos k_m h}, \text{ for } m = 1,2,... \end{cases} \tag{20}$$

$$f_n(y) = \begin{cases} \dfrac{\cosh p_0 (h+y)}{\cosh p_0 h}, \text{ for } n = 0 \\ \dfrac{\cos p_n (h+y)}{\cos p_n h}, \text{ for } n = -2,-1,1,2,... \end{cases} \tag{21}$$

with $p_{-2} = \alpha + i\beta$, $p_{-1} = \alpha - i\beta (\alpha, \beta > 0)$. Here $R, T, A_m (m = 1,2,...)$ and $B_n (n = -2,-1,1,2,...)$ are unknown constants to be determined to obtain the velocity potentials completely. Using the conditions of continuity of ϕ and $\partial\phi/\partial x$ across the line $x = 0$, we get two different relations for the above unknown coefficients, which are of the form:

$$(1+R)\psi_0(y) + \sum_{m=1}^{\infty} A_m \psi_m(y) = T f_0(y) + \sum_{\substack{n=-2 \\ n \neq 0}}^{\infty} B_n f_n(y) \tag{22}$$

$$\begin{aligned} &(1-R) ik_0 \psi_0(y) + \sum_{m=1}^{\infty} A_m k_m \psi_m(y) = \\ &T(ip_0) f_0(y) - \sum_{\substack{n=-2 \\ n \neq 0}}^{\infty} B_n p_n f_n(y) \end{aligned} \tag{23}$$

Truncating m and n in Eqs. (22)–(23) after N terms, we obtain two relations involving $(2N+4)$ unknowns. Therefore, we get a system of over-determined linear algebraic equations

$$\boldsymbol{Ax} = \boldsymbol{b} \tag{24}$$

where the entries of $\boldsymbol{A}$ are the coefficients of $R, A_1, ..., A_N, T, B_{-2}, B_{-1}, B_1, ..., B_N$ on each discretized points in the interval $(-h, 0)$ along the y-axis,

$$\boldsymbol{x} = \left[R, A_1, ..., A_N, T, B_{-2}, B_{-1}, B_1, ..., B_N\right]^{\mathrm{T}}$$

$$\boldsymbol{b} = \left[-\psi_0(y_1), -ik_0\psi_0(y_1), -\psi_0(y_2), -ik_0\psi_0(y_2), ...\right]^{\mathrm{T}}$$

with $y_1, y_2, ...,$ being infinite numbers of equally spaced discrete points in the interval $(-h, 0)$. In order to be able to utilize the method of least squares as well as the singular value decomposition method successfully, occurrence of ill-conditioned matrices must be avoided.

There are $(2N+4)$ unknowns in the above system which can be determined algebraically by the method of least squares and singular value decomposition.

3.2 Numerical results

We consider the numerical computation of the reflection and transmission coefficients due to the interaction of surface water waves with a semi-infinite thin elastic plate modelled as a thin ice-cover, which are calculated from the system of Eqs. (22)–(23), by using the method of least squares (LS) and singular value decomposition (SVD). In the following table, we present the variation of $|R|$ and $|T|$ for various values of N (number of terms in the evanescent wave modes for ϕ) and M (number of equally spaced discrete points of y). In this case, we fix the elastic parameters as $D/h^4 = 0.1$; $\varepsilon/h = 0.1$, the non-dimensional depth of fluid as 5 and the value of Kh as 0.03. It is known (Chakrabarti and Martha, 2009) that the energy-balance relation or the energy identity for the class of surface water wave problems involving a semi-infinite floating elastic plate on the surface of water of finite depth satisfies the following relation:

$$K_r^2 + K_d K_t^2 = 1 \tag{25}$$

where

$$K_d = \frac{k_0 \sinh 2k_0 h}{p_0 \sinh 2p_0 h} \times \left[\frac{2p_0 h (Dp_0^4 + 1 - \varepsilon K) + (5Dp_0^4 + 1 - \varepsilon K)\sinh 2p_0 h}{2k_0 h + \sinh 2k_0 h} \right]$$

$$K_r = |R|, \quad K_t = \left| \frac{T\, p_0 \tanh p_0 h}{k_0 \tanh k_0 h} \right|.$$

Table 1 Numerical values of $|R|$ and $|T|$ for various values of M and N

M	N	LS solution $\|R\|$	LS solution $\|T\|$	LS solution $K_r^2 + K_d K_t^2$	SVD solution $\|R\|$	SVD solution $\|T\|$	SVD solution $K_r^2 + K_d K_t^2$
10	2	7.7494e-04	0.9994	1.0001	7.7494e-04	0.9994	1.0001
10	3	7.7546e-04	0.9994	1.0001	7.7546e-04	0.9994	1.0001
13	2	7.7396e-04	0.9994	1.0001	7.7396e-04	0.9994	1.0001
13	3	7.7396e-04	0.9994	1.0001	7.7396e-04	0.9994	1.0001
17	4	7.7413e-04	0.9994	1.0001	7.7413e-04	0.9994	1.0001
25	5	7.7428e-04	0.9994	1.0001	7.7428e-04	0.9994	1.0001

In Table 1, the numerical values for $|R|, |T|$ and $K_r^2 + K_d K_t^2$, are given against different values of M and N for the numerical values of the roots $k_0 = 0.0795$, $k_1 = 3.1397$, $k_2 = 6.2822$, ..., $p_0 = 0.0796$, $p_1 = 3.1414$, $p_2 = 6.2832$,..., $p_{-2} = 1.2489 + 1.2565\mathrm{i}$ and $p_{-1} = 1.2489 - 1.2565\mathrm{i}$. By using the method of least squares and singular value decomposition, we are able to successfully achieve the satisfaction of the energy identity (25) almost accurately.

Figs. 1 and 2 show the plots of reflection and transmission coefficients for different values of the elastic parameter D/h^4. In both the figures, we fixed the non-dimensional depth of the fluid as 5; the number of terms in the evanescent wave modes for both plate covered and free surface regions N as 7; and the other elastic parameter ε/h as 0.1. From these figures, it is observed that when the values of elastic parameter increase, the reflection coefficient decreases while the transmission coefficient increases. That means as the values of the elastic parameter increase, the plate becomes rigid and most of the wave energy which concentrates near the covered region is transmitted below the plate and less amount of wave energy is reflected back by the plate.

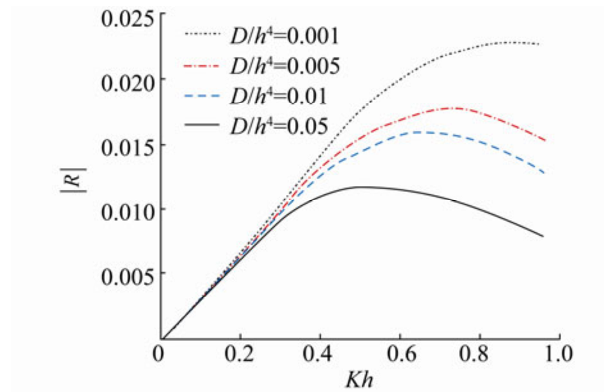

Fig. 1 Reflection coefficient $|R|$ plotted against Kh for $\varepsilon/h = 0.1$ and N=7

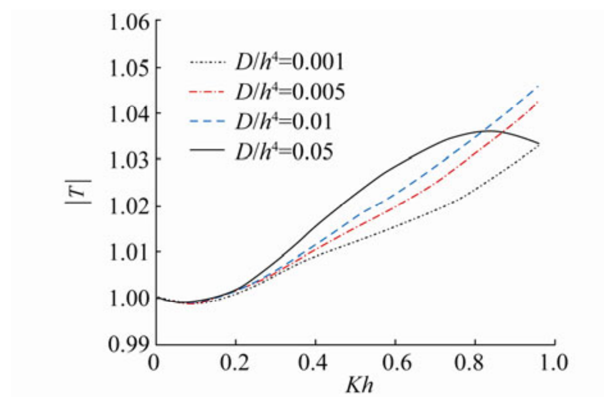

Fig. 2 Transmission coefficient $|T|$ plotted against Kh for $\varepsilon/h = 0.1$ and N=7

For long waves corresponding to a small incident wave number $k_0 h$ (i.e., for a small incident wave frequency Kh of surface waves), the fluid flow is uniform along the horizontal direction. As a result, there is little wave reflection by the horizontal plate for long waves. The plate appears to be transparent to the incident wave and the wave reflection disappears. In such a case, the reflection coefficient $|R|$ tends to zero and the transmission coefficient $|T|$ approaches to one, as the wave number of the open region $k_0 h$ vanishes (i.e., the incident wave frequency Kh of surface waves converges to zero), which means that a major part of the wave energy for long waves is transmitted into the covered region $(0 < x < \infty)$. On the other hand, when the wave number of the open region increases, that is, for the case of short waves corresponding to large wave number $k_0 h$, the reflection and transmission coefficients increase. This is due to the fact that for short waves, which correspond to large wave numbers, most of the wave energy is concentrated near the plate covered region. Thus, a large proportion of the wave energy is transmitted into the covered region and less amount of wave energy is reflected back by the plate. Thus, the incident waves of smaller wavelength penetrate farther into the plate-covered region.

4 Water wave scattering by a pair of semi-infinite elastic plates

In this section, we have considered the problem of scattering of surface water waves involving an ocean of finite depth having a flat rigid bottom, whereas on the upper surface of the ocean there exist a pair of thin uniform elastic plates (modelled as thin ice-plates) with a finite gap in between. In this case, different boundary conditions on the upper surface are satisfied on different sides of the finite free surface (gap), composed of two thin elastic plates floating on the surface. Time-harmonic waves of a particular frequency are assumed to propagate normally to the edges of the floating plates. Using the algebraic approaches involving the least squares and singular value decomposition methods, we have determined the reflection and transmission coefficients of the presently considered scattering problem, approximately.

We assume irrotational motion of an inviscid incompressible fluid (water) of finite depth under the action of gravity which is covered by two thin uniform semi-infinite elastic plates modelled as thin ice sheets, separated by a gap of finite width. The fluid is of infinite horizontal extent in x-direction while the depth is along y-direction which is considered vertically upwards with $y=0$ as the mean position of the top surface and $y=-h$ as the bottom surface. We further assume that all the motions are time harmonic with angular frequency ω. Under these assumptions, the velocity potential in the fluid can be written as $\mathrm{Re}\left[\phi(x,y)\mathrm{e}^{-\mathrm{i}\omega t}\right]$, where Re stands for the real part, and the potential function $\phi(x,y)$ satisfies Laplace's equation:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \quad \text{in} \quad -\infty<x<\infty, -h<y<0 \tag{26}$$

where the linearized boundary conditions on the top surface and bottom surface are:

$$\left(D_1\frac{\partial^4}{\partial x^4}+1-\varepsilon_1 K\right)\frac{\partial\phi}{\partial y}-K\phi=0, \text{ on } \quad y=0, -\infty<x<-a \tag{27}$$

$$\frac{\partial\phi}{\partial y}-K\phi=0 \quad \text{on} \quad y=0,\ -a<x<a \tag{28}$$

$$\left(D_2\frac{\partial^4}{\partial x^4}+1-\varepsilon_2 K\right)\frac{\partial\phi}{\partial y}-K\phi=0, \text{ on } \quad y=0,\ a<x<\infty \tag{29}$$

$$\begin{gathered}\frac{\partial^3\phi}{\partial x^2\partial y}\to 0,\ \frac{\partial^4\phi}{\partial x^3\partial y}\to 0\\ \text{as } y\to 0, x\to -a^+, \quad \text{and } y\to 0, x\to a^-\end{gathered} \tag{30}$$

$$\phi(x,y)\sim\begin{cases}\left(\mathrm{e}^{\mathrm{i}p_0^{(1)}x}+R\,\mathrm{e}^{-\mathrm{i}p_0^{(1)}x}\right)\dfrac{\cosh p_0^{(1)}(h+y)}{\cosh p_0^{(1)}h}, \\ \qquad \text{as } -\infty<x<-a, \\ \left(R_1\mathrm{e}^{-\mathrm{i}k_0x}+T_1\,\mathrm{e}^{\mathrm{i}k_0x}\right)\dfrac{\cosh k_0(h+y)}{\cosh k_0h} \\ \qquad \text{as } -a<x<a, \\ T_2\,\mathrm{e}^{\mathrm{i}p_0^{(2)}x}\dfrac{\cosh p_0^{(2)}(h+y)}{\cosh p_0^{(2)}h} \\ \qquad \text{as } a<x<\infty,\end{cases} \tag{31}$$

$$\frac{\partial\phi}{\partial y}=0 \quad \text{on} \quad y=-h,\ -\infty<x<\infty \tag{32}$$

where $K=\omega^2/g$; g the acceleration due to gravity. In the region $(-\infty<x<-a,-h<y<0), D_1=(L_1/\rho g)$; L_1 is the flexural rigidity of the elastic plate, $\varepsilon_1=\left(\rho_0^{(1)}/\rho\right)h_0$; $\rho_0^{(1)}$ is the density of the floating plate in this region and in the region $(a<x<\infty\ ,-h<y<0)$, $D_2=(L_2/\rho g)$; L_2 is the flexural rigidity of the elastic plate, $\varepsilon_2=\left(\rho_0^{(2)}/\rho\right)h_0$; $\rho_0^{(2)}$ is the density of the floating plate in this region, ρ is the density of water, h_0 is the very small thickness of both the elastic plates. The unknown complex constants R, R_1, T_1 and T_2 are related with the reflection and transmission coefficients; $k_0, p_0^{(1)}$ and $p_0^{(2)}$ are positive real numbers satisfying, respectively, the dispersion relations:

$$K-u\tanh uh=0 \tag{33}$$

$$\left(D_1v^4+1-\varepsilon_1K\right)v\tanh vh-K=0 \tag{34}$$

and

$$\left(D_2\gamma^4+1-\varepsilon_2K\right)\gamma\tanh\gamma h-K=0 \tag{35}$$

Now it may be noted here that the Eqs. (33), (34) and (35), respectively, represent the dispersion relations for the open region $(-a<x<a,-h<y<0)$ and the covered regions $(-\infty<x<-a,-h<y<0)$ and $(a<x<\infty,-h<y<0)$. In Eq. (33), there are two non-zero real roots $\pm k_0$ and a countable infinity of purely imaginary roots $\pm iu_m$ $(m=1,2,...)$. Similarly, in Eqs. (34) and (35), respectively, there are two non-zero real roots $\pm p_0^{(1)}$ and $\pm p_0^{(2)}$, two complex conjugate pair of roots $\pm\left(\alpha^{(1)}\pm i\beta^{(1)}\right)$ and

$\pm\left(\alpha^{(2)} \pm i\beta^{(2)}\right)$ with $\alpha^{(j)}, \beta^{(j)} > 0 \quad (j=1,2)$, and a countable infinity of purely imaginary roots $\pm \mathrm{i}\nu_n$ and $\pm i\gamma_n \, (n=1,2,\ldots)$.

4.1 Method of solution

The entire fluid domain is divided into three regions: the plate-covered region $(-\infty < x < -a, -h < y < 0)$, the open region $(-a < x < a, -h < y < 0)$ and the plate-covered region $(a < x < \infty, -h < y < 0)$. The velocity potential in the fluid region is expressed as

$$\phi(x,y) \sim \begin{cases} \left(\mathrm{e}^{\mathrm{i}p_0^{(1)}x} + R\,\mathrm{e}^{-\mathrm{i}p_0^{(1)}x}\right)\psi_0(y) + \\ \sum\limits_{\substack{n=-2\\ n\neq 0}}^{\infty} B_n \mathrm{e}^{p_n^{(1)}x} f_n^{(1)}(y), \text{ for } -\infty < x < -a, \\ \left(R_1 \mathrm{e}^{-\mathrm{i}k_0 x} + T_1\,\mathrm{e}^{\mathrm{i}k_0 x}\right)\psi_0(y) + \\ \sum\limits_{m=1}^{\infty} \begin{bmatrix} E_m \sinh k_m(x-a) \\ +F_m \sinh k_m(x+a) \end{bmatrix} \psi_m(y), \\ \qquad\qquad \text{for } -a < x < a, \\ T_2\,\mathrm{e}^{\mathrm{i}p_n^{(2)}x} f_0^{(2)}(y) + \sum\limits_{\substack{n=-2\\ n\neq 0}}^{\infty} A_n \mathrm{e}^{-p_n^{(2)}x} f_n^{(2)}(y), \\ \qquad\qquad \text{for } a < x < \infty, \end{cases} \tag{36}$$

where

$$\psi_m(y) = \begin{cases} \dfrac{\cosh k_0(h+y)}{\cosh k_0 h}, \text{ for } m=0 \\ \dfrac{\cos k_m(h+y)}{\cos k_m h}, \text{ for } m=1,2,\ldots \end{cases} \tag{37}$$

$$f_n^{(j)}(y) = \begin{cases} \dfrac{\cosh p_0^{(j)}(h+y)}{\cosh p_0^{(j)} h}, \text{ for } j=1,2,\, n=0 \\ \dfrac{\cos p_n^{(j)}(h+y)}{\cos p_n^{(j)} h}, \text{ for } j=1,2,\, n=-2,-1,1,2,\ldots \end{cases} \tag{38}$$

with $p_{-2}^{(j)} = \left(\alpha^{(j)} + \mathrm{i}\beta^{(j)}\right)$, $p_{-1}^{(j)} = \left(\alpha^{(j)} - \mathrm{i}\beta^{(j)}\right) \left(\beta^{(j)} > \alpha^{(j)} > 0\right)$. Here $R, R_1, T_1, T_2, A_n, B_n (n=-2,-1,1,2,\ldots)$ and $E_m, F_m \; (m=1,2,\ldots)$ are unknown constants to be determined to obtain the velocity potential completely.

Using the conditions of continuity of ϕ and $\partial\phi/\partial x$ across the line $x=-a$ and $x=a$, we get four different relations for the above unknown coefficients. Then truncating m and n after N terms, we obtain four relations involving $(4N+8)$ unknowns.

Therefore, we get a system of over-determined linear algebraic equations of the form

$$\boldsymbol{Ax} = \boldsymbol{b} \tag{39}$$

where the entries of $\boldsymbol{A}$ are the coefficients of $R, R_1, T_1, T_2, A_{-2}, A_{-1}, A_1, \ldots, A_N$, $B_{-2}, B_{-1}, B_1, \ldots, B_N, E_1, \ldots, E_N$, $F_1, \ldots, F_N$ on each discretized points in the interval $(-h, 0)$ along the y-axis,

$\boldsymbol{x} = \left[R, R_1, T_1, T_2, A_{-2}, A_{-1}, A_1, \ldots, A_N, B_{-2}, B_{-1}, B_1, \ldots, B_N, E_1, \ldots, E_N, F_1, \ldots, F_N\right]^{\mathrm{T}}$,

$\boldsymbol{b} = \left[G(y_1),\, 0,\, \mathrm{i}p_0^{(1)}G(y_1),\, 0,\, G(y_2),\, 0,\, \mathrm{i}p_0^{(1)}G(y_2),\, 0, \ldots\right]^{\mathrm{T}}$,

$G(y) = -\mathrm{e}^{-\mathrm{i}\,p_0^{(1)}a} f_0^{(1)}(y)$, and $y_1, y_2, \ldots$, are infinite numbers of discrete points in the interval $(-h, 0)$.

There are $(4N+8)$ unknowns in the above system which can be determined algebraically by the method of least squares and singular value decomposition.

4.2 Numerical results

We consider the numerical computation of the reflection and transmission coefficients due to the interaction of surface water waves with a pair of thin elastic plates separated by a gap of finite width, which are calculated from Eq. (39). In the following tables, we present the variation of $|R|, |R_1|, |T_1|$ and $|T_2|$ for various values of N (number of terms in the evanescent wave modes for ϕ and M (number of equally spaced discrete points of y). In this case, we fix the elastic parameters as $D_1/h^4 = 0.1$; $\varepsilon_1/h = 0.1$, $D_2/h^4 = 0.3$; $\varepsilon_2/h = 0.1$, the non-dimensional depth of fluid as 5 and the value of Kh as 0.03. It is known (Chakrabarti and Martha, 2009) that the energy-balance relation or the energy identity for the class of surface water wave problems involving a pair of floating elastic plates with a finite gap in between on the surface of water of finite depth satisfies the following relation:

$$K_r^{\,2} + K_d K_t^{\,2} = 1 \tag{40}$$

where

$$K_d = \frac{p_0^{(1)} \sinh 2p_0^{(1)}h}{p_0^{(2)} \sinh 2p_0^{(2)}h} \times \left[\frac{2p_0^{(2)}h(D_2 p_0^{(2)4} + 1 - \varepsilon_2 K) + (5D_2 p_0^{(2)4} + 1 - \varepsilon_2 K)\sinh 2p_0^{(2)}h}{2p_0^{(1)}h(D_1 p_0^{(1)4} + 1 - \varepsilon_1 K) + (5D_1 p_0^{(1)4} + 1 - \varepsilon_1 K)\sinh 2p_0^{(1)}h}\right] \tag{41}$$

$$K_r = |R|, \quad K_t = \left|\frac{T_2\, p_0^{(2)}\, \tanh p_0^{(2)}h}{p_0^{(1)}\, \tanh p_0^{(1)}h}\right| \tag{42}$$

Table 2 Numerical values of $|R|$, $|R_1|$ and $|T_1|$ and $|T_2|$ for various values of M and N (LS solution)

M	N	$\lvert R\rvert$	$\lvert R_1\rvert$	$\lvert T_1\rvert$	$\lvert T_2\rvert$	$K_r^{\,2} + K_d K_t^{\,2}$
8	2	2.6478733e-04	7.7457002e-04	1.0006008	1.0000058	1.0000252
8	3	1.8633604e-05	7.9385284e-04	1.0005842	1.0000063	1.0000261
10	2	3.2165429e-04	7.7280847e-04	1.0006098	1.0000061	1.0000258
10	3	2.3796480e-04	7.7292724e-04	1.0006117	1.0000064	1.0000264
15	4	2.8489279e-04	7.7419402e-04	1.0006034	1.0000061	1.0000258
22	4	2.8808999e-04	7.7393991e-04	1.0006052	1.0000063	1.0000261

Table 3 Numerical values of $|R|$, $|R_1|$ and $|T_1|$ and $|T_2|$ for various values of M and N (SVD solution)

M	N	$\lvert R\rvert$	$\lvert R_1\rvert$	$\lvert T_1\rvert$	$\lvert T_2\rvert$	$K_r^{\,2}+K_dK_t^{\,2}$
8	2	2.6478728e-04	7.7457003e-04	1.0006008	1.0000058	1.0000252
8	3	1.8103194e-05	7.9353593e-04	1.0005842	1.0000066	1.0000268
10	2	3.2165430e-04	7.7280848e-04	1.0006098	1.0000061	1.0000258
10	3	2.3797612e-04	7.7289804e-04	1.0006117	1.0000065	1.0000265
15	4	4.2955657e-04	6.0005575e-04	1.0007728	1.0003533	1.0007205
22	4	4.0613051e-04	7.6772135e-04	1.0006195	1.0000312	1.0000760

In the Tables 2 and 3, the numerical values for $|R|,|R_1|$, $|T_1|,|T_2|$ and $K_r^{\,2}+K_dK_t^{\,2}$ are given against different values of M and N for the numerical values of the roots $k_0=0.0795$, $k_1=3.1397$, $k_2=6.2822$, ..., $p_0^{(1)}=0.0795757$, $p_1^{(1)}=3.14141$, $p_2^{(1)}=6.28317$, ..., $p_{-2}^{(1)}=1.24891+1.25655\mathrm{i}$, $p_{-1}^{(1)}=1.24891-1.25655\mathrm{i}$, $p_0^{(2)}=0.0795753$, $p_1^{(2)}=3.141529$, $p_2^{(2)}=6.28318$, ..., $p_{-2}^{(2)}=0.947129+0.954802\mathrm{i}$ and $p_{-1}^{(2)}=0.947129-0.954802\mathrm{i}$. By using the method of least squares and singular value decomposition, we have been able to successfully achieve the satisfaction of the energy identity (40) almost accurately.

Figs. 3–6 show the plots of reflection and transmission coefficients due to a pair of floating semi-infinite elastic plates separated by a gap of finite width for different values of the elastic parameters D_1/h^4 and D_2/h^4. In all the figures, we fixed the non-dimensional depth of the fluid as 5, the number of terms in the evanescent wave modes for both the plate covered and open regions N as 4, and the value of the elastic parameters ε_1/h and ε_2/h as 0.1.

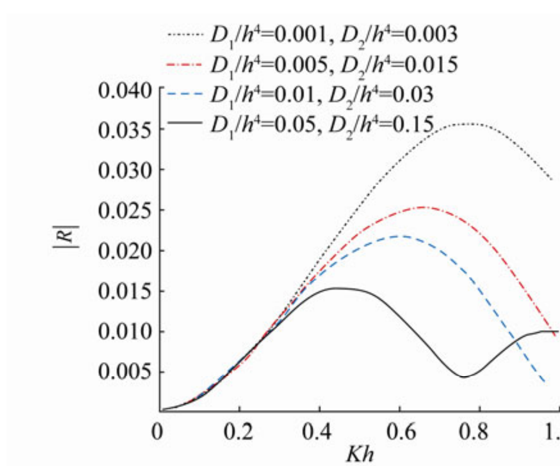

Fig. 3 Reflection coefficient $|R|$ plotted against *Kh* for $\varepsilon_1/h=\varepsilon_2/h=0.1$ and *N*=4

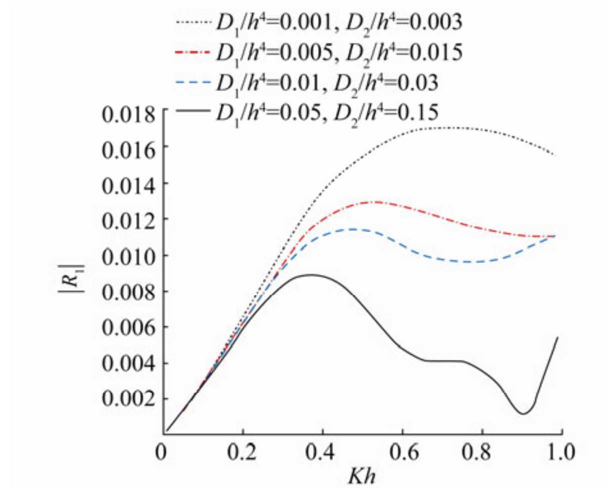

Fig. 4 Reflection coefficient $|R_1|$ plotted against *Kh* for $\varepsilon_1/h=\varepsilon_2/h=0.1$ and *N*=4

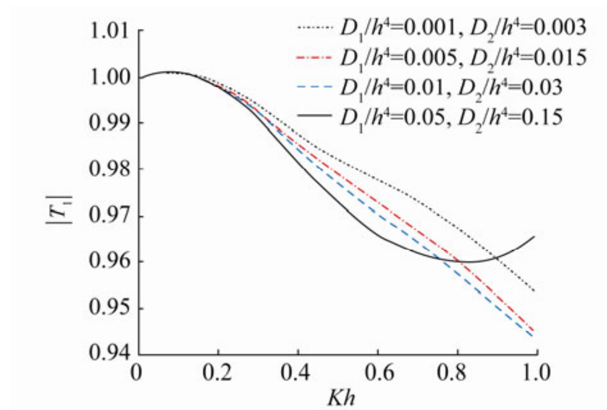

Fig. 5 Transmission coefficient $|T_1|$ plotted against *Kh* for $\varepsilon_1/h=\varepsilon_2/h=0.1$ and *N*=4

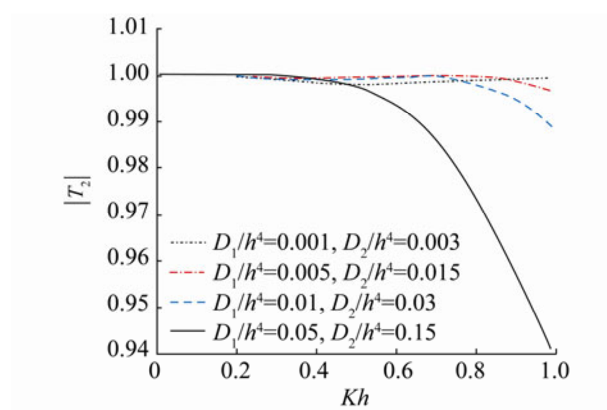

Fig. 6 Transmission coefficient $|T_2|$ plotted against *Kh* for $\varepsilon_1/h=\varepsilon_2/h=0.1$ and *N*=4

Figs. 3 and 6, respectively, show the reflection coefficients due to an incident wave in the plate covered region $(-\infty<x<-a)$ and the transmission coefficient due to an incident wave in the plate covered region

$(a < x < \infty)$, whereas Figs. 4 and 5, respectively, show the reflection and transmission coefficients due to an incident wave in the open region $(-a < x < a)$. From all these figures it is observed that as the value of elastic parameters increases, the plates become rigid and the peak values of reflection and transmission coefficients decrease. As in the case of wave scattering by a semi-infinite elastic plate, here also when the value of elastic parameters increases, most of the wave energy which concentrates near the covered region is transmitted below the plates and less amount of wave energy is reflected back by the plates. The values obtained for the reflection coefficient $|R|$ in the covered region $(-\infty < x < -a)$ are greater than those for the reflection coefficient $|R_1|$ in the open region $(-a < x < a)$. Moreover, the rate of change of the value of reflected energy is more for both the covered $(-\infty < x < -a)$ and open $(-a < x < a)$ regions.

It is observed from Figs. 5 and 6 that when the values of elastic parameters D_1/h^4 and D_2/h^4 increase, the rate of change of the values of the transmitted energy is very negligible for both open $(-a < x < a)$ and covered $(a < x < \infty)$ regions for certain values of Kh (i.e., for long waves corresponding to a small incident wave number $k_0 h$). In such a situation, the value of the transmitted energy decreases from one in the open region and approaches to one in the covered region $(a < x < \infty)$. This shows that these transmitted energies are somewhat insensitive to the changes in the elastic parameters of the plates and do not change appreciably when the elastic plates are approximated by a rigid lid.

5 Conclusions

A class of mixed boundary value problems arising in the problem of scattering of surface water waves involving an ocean of finite depth having a flat rigid bottom, with the upper surface of the ocean being bounded by a thin semi-infinite elastic plate and a pair of thin semi-infinite elastic plates, separated by a gap of finite width, are considered. In such a situation different waves can exist at different wave numbers for any given frequency. There are three kinds of waves acting below the plates: (i) an undamped progressive wave, (ii) evanescent wave modes and (iii) a decaying progressive wave, and two kinds of waves acting on the open region: (i) an undamped progressive wave and (ii) evanescent wave modes. By assuming two-dimensional linear water wave theory, the series solutions have been obtained in the respective regions by using the method of eigenfunction expansions. In both the problems, the numerical values of the reflection and transmission coefficients are obtained by solving an over-determined system of linear algebraic equations with the help of least squares and singular value decomposition method and depicted graphically against different values of wave numbers for different elastic parameters of the floating plates. Our main aim is to check the energy balance relations for the water waves scattering problems involving floating elastic plates by applying the present approach of solutions. The main advantage of the present approach of solutions that follows is that by using both the methods, the values of reflection and transmission coefficients are found here to satisfy the energy-balance relation associated with the corresponding scattering problem under consideration. It eliminates the need to use large and cumbersome analytical methods for the solutions of such type of problems. The present approach is much simpler than the method involving the Wiener-Hopf technique utilized by Tkacheva (2001) to solve these types of mixed boundary value problems in water wave theory.

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Appendix

Locating the roots of the dispersion equations

The dispersion equation for the ice-covered sea, given in Eq. (34), is

$$\Delta(v) \equiv \left(Dv^4 + 1 - \varepsilon K\right) v \tanh vh - K = 0 \qquad \text{(A.1)}$$

We can determine easily the real and pure imaginary roots by slight rearrangement of the equation (A.1). Any real root v must satisfy

$$\tanh vh = \frac{K}{\left(Dv^4 + 1 - \varepsilon K\right) v} \qquad \text{(A.2)}$$

When $(1 - \varepsilon K) > 0$, the polynomial term of right side of Eq. (A.2) is always positive and when $(1 - \varepsilon K) < 0$, the polynomial term is negative for $0 < v < \sqrt[4]{(\varepsilon K - 1)/D}$. In either case, the polynomial term and the hyperbolic tangent term intersect exactly once for $v > 0$, and since each function is odd in v there are two real roots occurring given by $\pm k_0$, with

$$k_0 > \begin{cases} 0, & \text{if } (1 - \varepsilon K) \geq 0 \\ \sqrt[4]{(\varepsilon K - 1)/D}, & \text{if } (1 - \varepsilon K) < 0 \end{cases} \qquad \text{(A.3)}$$

The lower bound given can be used as the starting point for a numerical root finding procedure.

When v is pure imaginary, i.e., $v = iv_n$ for some real v_n, the root satisfies

$$\tan v_n h = \frac{-K}{\left(D v_n^{\ 4} + 1 - \varepsilon K\right) v_n} \tag{A.4}$$

For the case $\left(1-\varepsilon K\right) > 0$, the polynomial term is always negative. It is clear that each branch of the tan function, except the branch that passes through the origin, intersects the polynomial term exactly once and hence the n th positive root satisfies $\left(n - 1/2\right)\pi < v_n h < n\pi$ $\left(n = 1, 2, ...\right)$, with $v_n h \to n\pi$ as $n \to \infty$. Again, since the functions are odd, roots also occur at $-v_n\left(n = 1, 2, ...\right)$. For the case $\left(1-\varepsilon K\right) < 0$, the polynomial term is positive for $v_n h < \sqrt[4]{\left(\varepsilon K - 1\right)/D}$ and we have the more general bound

$$\begin{aligned} &\left(n-1\right)\pi < v_n h < \left(n - \frac{1}{2}\right)\pi, \\ &\text{if } \sqrt[4]{\left(\varepsilon K - 1\right)/D} \geq \left(n - \frac{1}{2}\right)\frac{\pi}{h} \end{aligned} \tag{A.5}$$

$$\begin{aligned} &\left(n - \frac{1}{2}\right)\pi < v_n h < n\pi, \\ &\text{if } \sqrt[4]{\left(\varepsilon K - 1\right)/D} < \left(n - \frac{1}{2}\right)\frac{\pi}{h} \end{aligned} \tag{A.6}$$

These bounds provide an initial bracket suitable for initializing a numerical procedure to evaluate each purely imaginary root.

Now we describe the four remaining complex roots. Since $\Delta\left(z\right)$ is even and has real coefficients it follows that if z is a root, then so do $-z, \bar{z}$ and $-\bar{z}$. Note that z cannot be zero except possibly when $K = 0$. So we may take z to have positive real and imaginary parts.

Now we consider the function

$$f\left(z\right) = \left(Dz^4 + 1 - \varepsilon K\right) z \tan zh + K \tag{A.7}$$

Since zeros of $f\left(z\right)$ given in Eq. (A.7) are $\left(-i\right)$ times the zeros of $\Delta\left(z\right)$ given in equation (A.1), we find that $\left|\mathrm{Im}\left(z\right)\right| > \left|\mathrm{Re}\left(z\right)\right|$. These can be computed using fixed-point iteration schemes.

The dispersion equation for the open sea may also be analyzed in this way with the primary difference being that the fourth-order term in v that appears in Eq. (34) does not appear in Eq. (33) for u. Thus, the open sea dispersion equation has the same structure of real and pure imaginary roots as (34), but does not have the four extra complex roots.

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