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# Boundary Control of Coupled Nonlinear Three Dimensional Marine Risers

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**Abstract:** This paper presents a design of boundary controllers implemented at the top end for global stabilization of a marine riser in a three dimensional space under environmental loadings. Based on the energy approach, nonlinear partial differential equations of motion, including bending-bending and longitudinal-bending couplings for the risers are derived. The couplings cause mutual effects between the three independent directions in the riser's motions, and make it difficult to minimize its vibrations. The Lyapunov direct method is employed to design the boundary controller. It is shown that the proposed boundary controllers can effectively reduce the riser's vibration. Stability analysis of the closed-loop system is performed using the Lyapunov direct method. Numerical simulations illustrate the results.
**Keywords:** marine risers; boundary control; nonlinear dynamics; equations of motion; nonlinear couplings

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## 1 Introduction

In offshore petroleum production, marine risers are crucial in transporting petroleum products from wellheads to floating rigs, containing drill strings and carrying mud in drilling operations. The marine riser is subjected to environmental loading (waves, wind, and ocean currents), vortex induced vibrations and rig drifts, tension from the rig heave motion. In some cases these phenomena can reduce risers' lifespan and lead to an interruption of offshore operations. Furthermore, due to the high length to diameter ratio the riser's slender body makes its controlling and maintaining a challenging engineering task.

For the dynamic analysis purpose, the marine riser is considered as a distributed system which is modeled by a set of partial differential equations (PDE) and boundary conditions (Niedzwecki and Liagre, 2003). Dynamical systems governed by PDEs are difficult to control and have received a lot of attention. The most classical control strategy to the distributed systems was based on modal analysis (Balas 1978; Cavallo and De Maria, 1999; Fung and Liao, 1995). The modal analysis was used to derive a truncated model of the given system. Only some critical modes of the infinite dimensional and distributed parameter systems were observed and controlled. However, the control quality is substantially affected by observation and control spill-over due to residual (uncontrolled) modes. In addition, the requirement of arranging distributed actuators and sensors poses many difficulties in bringing the modal analysis-based control into practice. It might be problematic to deploy distributed devices in some cases, such as when controlling a deep-water riser.

In order to overcome aforementioned drawbacks of the modal analysis approach, a number of control methods have been developed to deal with the original PDE systems of the infinite dimensional systems instead of their truncated models. In Ge *et al*. (2001), the variable structure control was employed to regulate a flexible beam. The control design process was directly based on the PDE equations of motion. However, it is difficult to generalize the design procedure to other flexible systems. An elegant boundary control design can be found in Krstic *et al*. (2006a, 2006b, 2007). The authors successfully established an integral transformation to convert a beam system into a target system, with known dynamical responses. The main aim of the transformation was to find a proper gain kernel, then perform an inverse transformation, which has a tendency to be a very complicated task due to the systems complexities. Based on Lyapunov's direct method, various boundary controllers have been proposed for flexible string-like and beam-like systems. Boundary control of different string models was developed by Shahruz and Narasimha (1997), Shahruz and Kurmaji (1997), Shahruz and Krishna (1996), Kim and Jung (2011). It is shown that with simple boundary feedbacks, exponential stability can be achieved. In Fard and Sagatun (2001) and Queiroz *et al*. (2000), boundary control was used for stabilizing string and beam systems. Tanaka and Iwanmoto (2007) developed an active boundary control that could add two additional boundary conditions to four already well-known boundary conditions. The aim of this research was to produce a vibration-free state for an Euler-Bernoulli beam system. Although the proposed control method was named an active boundary control, the approach presented in the paper was actually searching for a distributed control law acting on any designated area of the system. Consequently,

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this should be considered as a branch of the modal control and it would be affected by spill-over instability. Due to the systematic approach and the ease of implementation in practice, applications of boundary control in marine riser vibration suppression have received increased attention. In Do and Pan (2008), How *et al*. (2009), Ge *et al*. (2010), He *et al*. (2011), boundary controllers were proposed for controlling vibration of marine risers based on the Lyapunov direct method. In Do and Pan (2008), a riser-actuator dynamics were taken into account, whereas, in He *et al*. (2011) a marine riser with vessel dynamics was considered. An interesting work on controlling marine risers was presented in Do and Pan (2009) where a boundary controller for a coupled system consisting of a three-dimensional riser and boundary actuators. The riser model was suitable for a class of flexible risers since the riser is modeled as a rod-type system and not a beam-type system. In the aforementioned references, coupled dynamics such as bending-bending and longitudinal-bending effects were not entirely considered, the riser motions were restricted in one plane. The ignorance of coupling can directly deteriorate the performance of the controlled system. Therefore, it is necessary to include the couplings in the riser dynamics in the control design process.

In this paper, a global stabilization problem for three dimensional flexible marine risers under environmental disturbances is investigated. Equations of motion of the riser is described by a set of PDEs and boundary conditions derived by the energy approach. The riser dynamics possess some high nonlinearities due to the system couplings. The couplings show the direct effects between motions in three directions, and lead to a complex control design process. Based on Lyapunov's direct method, a boundary controller at the top end of the riser is designed. The proof of existence, uniqueness, and convergence of the solutions of the closed-loop system is provided. The proposed boundary controller in this paper guarantees that when there are no environmental disturbances, the riser is globally exponentially stabilized at its equilibrium position and that when the disturbances are presented, the riser is stabilized in the neighborhood of its equilibrium position.

## 2 Mathematical model

In deriving equations of motion of the riser, we assume that:

**Assumption**

1) The riser can be modeled as a beam because of its high length-to-diameter ratio.

2) Plane sections remain plane after deformation, i.e., warping is neglected.

3) The riser is locally stiff, i.e. cross-sections do not deform and the Poisson effect is neglected.

4) The riser material is homogeneous, isotropic and linearly elastic, i.e., it obeys Hookes's law.

5) Torsional and distributed moments induced by environmental disturbances are neglected.

6) The riser deforms in three dimensions.

7) Ball joints are placed at the both ends of the riser, i.e., there is no bending at the both ends.

8) Environmental disturbances are bounded.

**Remark** Items 1-4 imply that the riser will be modeled as a Bernoulli beam rather than a Timoshenko beam, and that the riser's extension is small. Bernoulli-Euler models are adequate for modeling the low frequency responses of beams. Item 5 indicates that fluid/gas transportation risers, rather than drilling risers, are considered and that moments induced by asymmetrically relative flow due to vortex shedding are ignored. Items 7 and 8 always hold in practice.

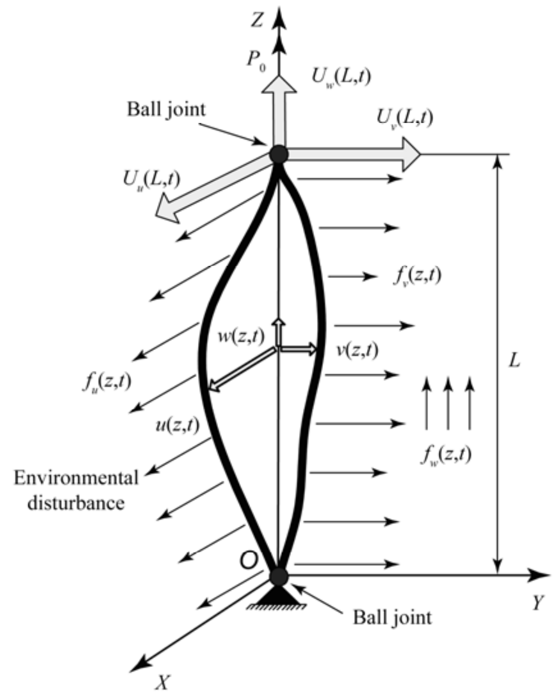



**Fig. 1 Riser coordinates**

The kinetic energy of the riser is given by

$$T = \frac{m_0}{2}\int_0^L \left[\left(\frac{\partial u(z,t)}{\partial t}\right)^2 + \left(\frac{\partial v(z,t)}{\partial t}\right)^2 + \left(\frac{\partial w(z,t)}{\partial t}\right)^2\right] \mathrm{d}z \quad (1)$$

where $u(z, t)$ and $v(z, t)$ are transverse displacements in the $X$ and $Y$ directions, respectively, and $w(z, t)$ is longitudinal displacement in the $Z$ direction. $L$ is the length of the riser, $m_0 = \rho A$ is the oscillating mass of the riser per unit length, $A$ is the riser cross section area, and $\rho$ is the mass density of the riser. It is assumed that the riser under consideration is subject to a positive constant tension $P_0$. The potential energy of the riser can be expressed as follows:

$$\begin{aligned} P = {} & \frac{EI}{2}\int_0^L \left[\left(\frac{\partial^2 u(z,t)}{\partial z^2}\right)^2 + \left(\frac{\partial^2 v(z,t)}{\partial z^2}\right)^2\right] \mathrm{d}z + \\ & \frac{p_0}{2}\int_0^L \left[\left(\frac{\partial u(z,t)}{\partial z}\right)^2 + \left(\frac{\partial v(z,t)}{\partial z}\right)^2\right] \mathrm{d}z + \\ & \frac{EA}{2}\int_0^L \left[\frac{\partial w(z,t)}{\partial z} + \frac{1}{2}\left(\frac{\partial u(z,t)}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial w(z,t)}{\partial z}\right)^2\right]^2 \mathrm{d}z \end{aligned} \quad (2)$$

where $E$ is the Young's modulus, $I$ is the second moment of the riser's cross section area. The first part in the potential energy expression quantifies bending, the second part is due to tension force and the third term is strain energy of the riser.

The work done by environmental disturbances acting on the riser is given by

$$W_f = \int_0^L f_u(z,t)u(z,t)\mathrm{d}z + \int_0^L f_v(z,t)v(z,t)\mathrm{d}z + \int_0^L f_w(z,t)w(z,t)\mathrm{d}z \tag{3}$$

where $f_u(z, t)$, $f_v(z, t)$ and $f_w(z, t)$ are the hydrodynamics forces acting on the riser in the $X$, $Y$, and $Z$ directions, respectively. The hydrodynamic forces can be given as Do and Pan (2008):

$$\begin{aligned}
f_u(z,t) &= f_{uD} + f_{uL}\\
f_v(z,t) &= f_{vD} + f_{vL}\\
f_v(z,t) &= f_{wD} + f_{wL}\\
f_{uD} &= -\Omega_{1D}u_t(z,t)\\
\Omega_{1D} &= \left(c_1 + C_D\frac{\rho_w D}{2}\sqrt{\frac{8}{\pi}}\sigma_u\right)\\
f_{vD} &= -\Omega_{2D}v_t(z,t)\\
\Omega_{2D} &= \left(c_2 + C_D\frac{\rho_w D}{2}\sqrt{\frac{8}{\pi}}\sigma_v\right)\\
f_{wD} &= -\Omega_{3D}v_t(z,t)\\
\Omega_{3D} &= \left(c_3 + C_D\frac{\rho_w D}{2}\sqrt{\frac{8}{\pi}}\left(\sigma_u + \sigma_v\right)\right)\\
f_{uL} &= C_M\frac{\rho_w \pi D^2 u_{1t}(z,t)}{4} + C_D\frac{\rho_w D}{2}\sqrt{\frac{8}{\pi}}\sigma_u(z,t)u_1(z,t)\\
f_{vL} &= C_M\frac{\rho_w \pi D^2 u_{2t}(z,t)}{4} + C_D\frac{\rho_w D}{2}\sqrt{\frac{8}{\pi}}\sigma_v(z,t)u_2(z,t)\\
f_{vL} &= C_L\frac{\rho_w D}{2}\sqrt{\frac{8}{\pi}}\left[\sigma_u(z,t)u_1(z,t) + \sigma_v(z,t)u_2(z,t)\right]
\end{aligned} \tag{4}$$

where $f_{uD}$, $f_{vD}$, $f_{wD}$ and $f_{uL}$, $f_{vL}$, $f_{wL}$ represent the distributed damping and external forces, $c_1$, $c_2$, and $c_3$ are the linear viscous damping coefficients, $\rho_w$ is the water density, $C_M$ is the acceleration drag coefficient, $C_D$ is the velocity drag coefficient, $C_L$ is the lift force coefficient, $D$ is the riser diameter, $\sigma_u(z, t)$ and $\sigma_v(z, t)$ are the root-mean-square of the water particle velocities, and $u_{1t}(z, t)$ and $u_{2t}(z, t)$ are the water particle accelerations in the $X$ and $Y$ directions, respectively. It is noted that in (4), the quadratic term due to the relative water velocities $u_t(z, t)-u_1(z, t)$ and $v_t(z, t)-u_2(z, t)$ are approximated by a linear expression involving the root mean square of the relative velocities, and the relative velocities are approximated by the water velocities $u_1(z, t)$ and $u_2(z, t)$.

The work done by active boundary actuators is

$$W_m = U_u(L,t)u(L,t) + U_v(L,t)v(L,t) + U_w(L,t)w(L,t) \tag{5}$$

where $U_u(L, t)$, $U_v(L, t)$ and $U_w(L, t)$ are the boundary control forces. The total work done on the system is presented as

$$\begin{aligned}
W &= \int_0^L f_u(z,t)u(z,t)\mathrm{d}z + \int_0^L f_v(z,t)v(z,t)\mathrm{d}z +\\
&\int_0^L f_w(z,t)w(z,t)\mathrm{d}zz + U_u(L,t)u(L,t) +\\
&U_v(L,t)v(L,t) + U_w(L,t)w(L,t)
\end{aligned} \tag{6}$$

The extended Hamilton's principle is given by

$$\int_{t_1}^{t_2} \delta(T - P + W)\mathrm{d}t = 0 \tag{7}$$

From this point onward, the argument $(z, t)$ is omitted whenever it is not confusing. Using integration by parts, the variation in the kinetic energy can be written as

$$\int_{t_1}^{t_2} \delta T\mathrm{d}z = -m_0\int_{t_1}^{t_2}\int_0^L\left(\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 v}{\partial t^2}\delta v + \frac{\partial^2 w}{\partial t^2}\delta w\right)\mathrm{d}z\mathrm{d}t \tag{8}$$

where $\delta_u = \delta_v = \delta_w = 0$ at $t = t_1$, $t_2$ have been used. In addition, the variation in the potential energy is given as

$$\begin{aligned}
&\delta P = EI\frac{\partial^2 u}{\partial z^2}\frac{\partial}{\partial z}\delta u\Big|_0^L - EI\frac{\partial^3 u}{\partial z^3}\delta u\Big|_0^L +\\
&EI\int_0^L\frac{\partial^4 u}{\partial z^4}\delta u\mathrm{d}z + EI\frac{\partial^2}{\partial z^2}\frac{\partial}{\partial z}\delta u\Big|_0^L - EI\frac{\partial^3 v}{\partial z^3}\delta v\Big|_0^L +\\
&EI\int_0^L\frac{\partial^4 v}{\partial z^4}\delta v\mathrm{d}z + P_0\frac{\partial u}{\partial z}\delta u\Big|_0^L -\\
&P_0\int_0^L\frac{\partial^2 u}{\partial z^2}\delta u\mathrm{d}z + P_0\frac{\partial v}{\partial z}\delta v\Big|_0^L - P_0\int_0^L\frac{\partial^2 v}{\partial z^2}\delta v\mathrm{d}z +\\
&EA\frac{\partial w}{\partial z}\delta w\Big|_0^L - EA\int_0^L\frac{\partial^2 w}{\partial z^2}\delta w\mathrm{d}z +\\
&\frac{EA}{2}\left(\frac{\partial u}{\partial z}\right)^3\delta u\Big|_0^L - \frac{3EA}{2}\int_0^L\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}\delta u\mathrm{d}z +\\
&\frac{EA}{2}\left(\frac{\partial v}{\partial z}\right)^3\delta v\Big|_0^L - \frac{3EA}{2}\int_0^L\left(\frac{\partial v}{\partial z}\right)^2\frac{\partial^2 v}{\partial z^2}\delta v\mathrm{d}z +\\
&\frac{EA}{2}\left(\frac{\partial u}{\partial z}\right)^2\delta w\Big|_0^L - EA\int_0^L\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\delta w\mathrm{d}z +\\
&EA\frac{\partial w}{\partial z}\frac{\partial u}{\partial z}\delta u\Big|_0^L - EA\int_0^L\frac{\partial^2 w}{\partial z^2}\frac{\partial u}{\partial z}\delta\mathrm{d}z -\\
&EA\int_0^L\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}\mathrm{d}z + \frac{EA}{2}\left(\frac{\partial v}{\partial z}\right)^2\delta w\Big|_0^L - EA\int_0^L\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}\delta w\mathrm{d}z +\\
&EA\frac{\partial w}{\partial z}\frac{\partial v}{\partial z}\delta v\Big|_0^L - EA\int_0^L\frac{\partial^2 w}{\partial z^2}\frac{\partial v}{\partial z}\delta v\mathrm{d}z - EA\int_0^L\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}\delta\mathrm{d}z +\\
&\frac{EA}{2}\frac{\partial u}{\partial z}\left(\frac{\partial v}{\partial z}\right)^2\delta u\Big|_0^L - \frac{EA}{2}\int_0^L\left[\frac{\partial^2 u}{\partial z^2}\left(\frac{\partial v}{\partial z}\right)^2 + \frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}\frac{\partial u}{\partial z}\right]\delta u\mathrm{d}z +\\
&\frac{EA}{2}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)^2\delta v\Big|_0^L - \frac{EA}{2}\int_0^L\left[\frac{\partial^2 v}{\partial z^2}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\frac{\partial v}{\partial z}\right]\delta v\mathrm{d}z
\end{aligned} \tag{9}$$

Substituting (1), (2) and (6) into (7) and using (8) and (9) results in

$$\int_{t_1}^{t_2}\int_0^L\left\{\left[\left(-m_0\frac{\partial^2 u}{\partial t^2}-EI\frac{\partial^4 u}{\partial z^4}+P_0\frac{\partial^2 u}{\partial z^2}+\frac{3EA}{2}\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+EA\frac{\partial^2 w}{\partial z^2}\frac{\partial u}{\partial z}+EA\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{EA}{2}\frac{\partial^2 u}{\partial z^2}\left(\frac{\partial v}{\partial z}\right)^2+EA\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}+f_u\right)\delta u+\left(-m_0\frac{\partial^2 v}{\partial t^2}-EI\frac{\partial^4 v}{\partial t^4}+P_0\frac{\partial^2 v}{\partial t^2}+\frac{3EA}{2}\left(\frac{\partial v}{\partial z}\right)^2\frac{\partial^2 v}{\partial z^2}+EA\frac{\partial^2 w}{\partial t^2}\frac{\partial v}{\partial z}+EA\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}+\frac{EA}{2}\frac{\partial^2 v}{\partial z^2}\left(\frac{\partial u}{\partial z}\right)^2+EA\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial z^2}+f_v\right)\delta u+\left(EA\frac{\partial^2 w}{\partial t^2}-EA\frac{\partial^2 w}{\partial z^2}+EA\frac{\partial u}{\partial z}+EA\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}+f_w\right)\delta u\right]\mathrm{d}z-EI\frac{\partial^2 u}{\partial z^2}\frac{\partial}{\partial z}\delta u\Big|_0^L+\left(EI\frac{\partial^3 u}{\partial z^3}-P_0\frac{\partial u}{\partial z}-\frac{EA}{2}\left(\frac{\partial u}{\partial z}\right)^3-EA\frac{\partial w}{\partial z}\frac{\partial u}{\partial z}-\frac{EA}{2}\frac{\partial u}{\partial z}\left(\frac{\partial v}{\partial z}\right)^2\right)\delta u\Big|_0^L-EI\frac{\partial^2 v}{\partial z^2}\frac{\partial}{\partial z}\delta v\Big|_0^L+\left(EI\frac{\partial^3 v}{\partial z^3}-P_0\frac{\partial v}{\partial z}-\frac{EA}{2}\left(\frac{\partial v}{\partial z}\right)^3-EA\frac{\partial w}{\partial z}\frac{\partial v}{\partial z}-\frac{EA}{2}\frac{\partial v}{\partial z}\left(\frac{\partial u}{\partial z}\right)^2\right)\delta v\Big|_0^L+EA\frac{\partial w}{\partial z}\delta w\Big|_0^L+\left(-EA\frac{\partial w}{\partial z}-\frac{EA}{2}\left(\frac{\partial u}{\partial z}\right)^2-\frac{EA}{2}\left(\frac{\partial v}{\partial z}\right)^2\right)\delta w\Big|_0^L\right\}\mathrm{d}t+U_u\delta u(L,t)+U_v\delta v(L,t)+U_w\delta w(L,t)=0 \tag{10}$$

Since $\delta u$, $\delta v$, and $\delta w$ are arbitrary over the domain $0 < z < L$, Eq. (10) holds provided that

$$-m_0\frac{\partial^2 u}{\partial t^2}-EI\frac{\partial^4 u}{\partial z^4}+P_0\frac{\partial^2 u}{\partial z^2}+\frac{3EA}{2}\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+EA\frac{\partial^2 w}{\partial z^2}\frac{\partial u}{\partial z}+EA\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{EA}{2}\frac{\partial^2 u}{\partial z^2}\left(\frac{\partial v}{\partial z}\right)^2+EA\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}+f_u=0 \tag{11}$$

$$-m_0\frac{\partial^2 v}{\partial t^2}-EI\frac{\partial^4 v}{\partial t^4}+P_0\frac{\partial^2 v}{\partial t^2}+\frac{3EA}{2}\left(\frac{\partial v}{\partial z}\right)^2\frac{\partial^2 v}{\partial z^2}+EA\frac{\partial^2 w}{\partial t^2}\frac{\partial v}{\partial z}+EA\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}+\frac{EA}{2}\frac{\partial^2 v}{\partial z^2}\left(\frac{\partial u}{\partial z}\right)^2+EA\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial z^2}+f_v \tag{12}$$

and

$$EA\frac{\partial^2 w}{\partial t^2}-EI\frac{\partial^2 w}{\partial z^2}+EA\frac{\partial u}{\partial z}+EA\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}+f_w=0 \tag{13}$$

for all $z\in[0, L]$, $t\in\mathbb{R}^+$. In addition,

$$-EI\frac{\partial^3 u}{\partial z^3}+P_0\frac{\partial u}{\partial z}+\frac{EA}{2}\left(\frac{\partial u}{\partial z}\right)^3+EA\frac{\partial w}{\partial z}\frac{\partial u}{\partial z}+\frac{EA}{2}\frac{\partial u}{\partial z}\left(\frac{\partial v}{\partial z}\right)^2=U_u \quad \text{at} \quad z=0,L \tag{14}$$

or

$$u=0 \quad \text{at} \quad z=0,L \tag{15}$$

and

$$-EI\frac{\partial^3 v}{\partial z^3}+P_0\frac{\partial v}{\partial z}+\frac{EA}{2}\left(\frac{\partial v}{\partial z}\right)^3+EA\frac{\partial w}{\partial z}\frac{\partial v}{\partial z}+\frac{EA}{2}\frac{\partial v}{\partial z}\left(\frac{\partial u}{\partial z}\right)^2=U_v \quad \text{at} \quad z=0,L \tag{16}$$

or

$$v=0 \quad \text{at} \quad z=0,L \tag{17}$$

and

$$EA\frac{\partial w}{\partial z}+\frac{EA}{2}\left(\frac{\partial u}{\partial z}\right)^2+\frac{EA}{2}\left(\frac{\partial v}{\partial z}\right)^2=U_w \quad \text{at} \quad z=0,L \tag{18}$$

or

$$w=0 \quad \text{at} \quad z=0,L \tag{19}$$

and

$$EI\frac{\partial^2 u}{\partial z^2}=0 \quad \text{at} \quad z=0,L \tag{20}$$

or

$$\frac{\partial u}{\partial z}=0 \quad \text{at} \quad z=0,L \tag{21}$$

and

$$EI\frac{\partial^2 v}{\partial z^2}=0 \quad \text{at} \quad z=0,L \tag{22}$$

or

$$\frac{\partial v}{\partial z}=0 \quad \text{at} \quad z=0,L \tag{23}$$

For the riser under consideration, ball joints placed at both ends imply that there is no bending at both ends, see Fig. 1. In addition, the lower end is fixed. Substituting the hydrodynamics forces in (4) results in the following riser dynamics

$$\begin{aligned}
&-m_0u_{tt}-EIu_{zzzz}+P_0u_{zz}+\frac{3EA}{2}u_z^2u_{zz}+EAw_{zz}u_z+EAw_zu_{zz}+\frac{EA}{2}u_{zz}v_z^2+EAv_zv_{zz}u_z-\varOmega_{1D}u_t+f_u=0,\\
&-m_0v_{tt}-EIv_{zzzz}+P_0v_{zz}+\frac{3EA}{2}v_z^2v_{zz}+EAw_{zz}v_z+EAw_zv_{zz}+\frac{EA}{2}v_{zz}u_z^2+EAu_zu_{zz}v_z-\varOmega_{2D}v_t+f_v=0,\\
&-m_0w_{tt}-EAw_{zz}+EAu_zu_{zz}+EAv_zv_{zz}-\varOmega_{3D}w_t+f_w=0,\\
&-EIu_{zzz}(L,t)+P_0u_z(L,t)+\frac{EA}{2}u_z^3(L,t)+EAw_z(L,t)u_z(L,t)+\frac{EA}{2}u_z(L,t)v_z^2(L,t)=U_u(L,t),\\
&-EIv_{zzz}(L,t)+P_0v_z(L,t)+\frac{EA}{2}v_z^3(L,t)+EAw_z(L,t)v_z(L,t)+\frac{EA}{2}v_z(L,t)u_z^2(L,t)=U_v(L,t),\\
&EAw_z(L,t)+\frac{EA}{2}u_z^2(L,t)+\frac{EA}{2}v_z^2(L,t)=U_w(L,t),\\
&u_{zz}(L,t)=v_{zz}(L,t)=u_{zz}(0,t)=v_{zz}(0,t)=0,\\
&u(0,t)=v(0,t)=w(0,t)=0
\end{aligned} \tag{24}$$

where the following notations $\frac{\partial(\bullet)}{\partial z}=(\bullet)_z$, $\frac{\partial^2(\bullet)}{\partial z^2}=(\bullet)_{zz}$, $\frac{\partial^3(\bullet)}{\partial z^3}=(\bullet)_{zzz}, \frac{\partial^4(\bullet)}{\partial z^4}=(\bullet)_{zzzz}, \frac{\partial(\bullet)}{\partial t}=(\bullet)_t$, and $\frac{\partial^2(\bullet)}{\partial t^2}=(\bullet)_{tt}$ have been used.

## 3 Boundary control design

**Control objectives** Subject to Assumption 1, design the boundary control forces $U_u(t)$, $U_v(t)$, and $U_w(t)$ from information at the top end of the riser for the riser system (24) to stabilize the riser at the equilibrium position, and:

1) In the case where disturbances $f_u$, $f_v$, and $f_w$ are ignored, $|u(z,t)|$, $|v(z,t)|$, $|w(z,t)|$, $\int_0^L u_z(z,t)\mathrm{d}z$, $\int_0^L v_z(z,t)\mathrm{d}s$, $\int_0^L w_z(z,t)\mathrm{d}z$, $\int_0^L u_t(z,t)\mathrm{d}z$, $\int_0^L v_t(z,t)\mathrm{d}z$, $\int_0^L w_t(z,t)\mathrm{d}z$, $\int_0^L u_{zz}\mathrm{d}z$, and $\int_0^L v_{zz}\mathrm{d}z$ exponentially converge to zero $\forall z\in[0,L]$ and $\forall t\geq t_0$.

2) In the case where disturbances $f_u$, $f_v$, and $f_w$ are present, $|u(z,t)|$, $|v(z,t)|$, $|w(z,t)|$, $\int_0^L u_z(z,t)\mathrm{d}z$, $\int_0^L v_z(z,t)\mathrm{d}s$, $\int_0^L u_t(z,t)\mathrm{d}z$, $\int_0^L w_z(z,t)\mathrm{d}z$, $\int_0^L v_t(z,t)\mathrm{d}z$, $\int_0^L w_t(z,t)\mathrm{d}z$, $\int_0^L u_{zz}\mathrm{d}z$, and $\int_0^L v_{zz}\mathrm{d}z$ exponentially converge to positive constants $\forall z\in[0,L]$ and $\forall t\geq t_0$.

Consider the following Lyapunov candidate function

$$
\begin{aligned}
v=&\frac{m_0}{2}\int_0^L\left(u_t^2+v_t^2+w_t^2\right)\mathrm{d}z+\frac{P_0}{2}\int_0^L\left(u_z^2+v_z^2\right)\mathrm{d}z+\\
&\frac{EA}{2}\int_0^L\left(w_z+\frac{u_z^2}{2}+\frac{v_z^2}{2}\right)^2\mathrm{d}z+\frac{EI}{2}\int_0^L\left(u_{zz}^2+v_{zz}^2\right)\mathrm{d}z+\\
&\rho_1\int_0^L uu_t\mathrm{d}z+\rho_2\int_0^L vv_t\mathrm{d}z+\rho_3\int_0^L ww_t\mathrm{d}z+\\
&\left(k_1+\frac{k_2\rho_1}{m_0}\right)u^2(L,t)+\left(k_3+\frac{k_4\rho_2}{m_0}\right)v^2(L,t)+\\
&\left(k_5+\frac{k_6\rho_3}{m_0}\right)w^2(L,t)
\end{aligned}\tag{25}
$$

Since $\forall t\geq 0$ we can write

$$
\begin{aligned}
&-\Upsilon_1\rho_1\int_0^L u^2\mathrm{d}z-\frac{\rho_1}{\Upsilon_1}\int_0^L u_t^2\mathrm{d}z\leq\rho_1\int_0^L uu_t\mathrm{d}z\leq\\
&\Upsilon_1\rho_1\int_0^L u^2\mathrm{d}z+\frac{\rho_1}{\Upsilon_1}\int_0^L u_t^2\mathrm{d}z
\end{aligned}\tag{26}
$$

$$
\begin{aligned}
&-\Upsilon_2\rho_2\int_0^L v^2\mathrm{d}z-\frac{\rho_2}{\Upsilon_2}\int_0^L v_t^2\mathrm{d}z\leq\rho_2\int_0^L vv_t\mathrm{d}z\leq\\
&\Upsilon_2\rho_2\int_0^L v^2\mathrm{d}z+\frac{\rho_2}{\Upsilon_2}\int_0^L v_t^2\mathrm{d}z
\end{aligned}\tag{27}
$$

$$
\begin{aligned}
&-\Upsilon_3\rho_3\int_0^L w^2\mathrm{d}z-\frac{\rho_3}{\Upsilon_3}\int_0^L w_t^2\mathrm{d}z\leq\rho_3\int_0^L ww_t\mathrm{d}z\leq\\
&\Upsilon_3\rho_3\int_0^L w^2\mathrm{d}z+\frac{\rho_3}{\Upsilon_3}\int_0^L w_t^2\mathrm{d}z
\end{aligned}\tag{28}
$$

where $\Upsilon_1$, $\Upsilon_2$, and $\Upsilon_3$ are positive constants. Since $u(0,t)=v(0,t)=w(0,t)=0$, an application of (A1) shows that the following inequalities hold

$$\Upsilon_1\rho_1\int_0^L u^2\mathrm{d}z\leq 4L^2\Upsilon_1\rho_1\int_0^L u_z^2\mathrm{d}z\tag{29}$$

$$\Upsilon_2\rho_2\int_0^L v^2\mathrm{d}z\leq 4L^2\Upsilon_2\rho_2\int_0^L v_z^2\mathrm{d}z\tag{30}$$

$$\Upsilon_3\rho_3\int_0^L w^2\mathrm{d}z\leq 4L^2\Upsilon_2\rho_3\int_0^L w_z^2\mathrm{d}z\tag{31}$$

Now, (26), (27), and (28) can be modified as

$$
\begin{aligned}
&-4L^2\Upsilon_1\rho_1\int_0^L u_z^2\mathrm{d}z-\frac{\rho_1}{\Upsilon_1}\int_0^L u_t^2\mathrm{d}z\leq\rho_1\int_0^L uu_t\mathrm{d}z\leq\\
&4L^2\Upsilon_1\rho_1\int_0^L u_z^2\mathrm{d}z+\frac{\rho_1}{\Upsilon_1}\int_0^L u_t^2\mathrm{d}z
\end{aligned}\tag{32}
$$

$$
\begin{aligned}
&-4L^2\Upsilon_2\rho_2\int_0^L v_z^2\mathrm{d}z-\frac{\rho_2}{\Upsilon_2}\int_0^L v_t^2\mathrm{d}z\leq\rho_2\int_0^L vv_t\mathrm{d}z\leq\\
&4L^2\Upsilon_2\rho_2\int_0^L v_z^2\mathrm{d}z+\frac{\rho_2}{\Upsilon_2}\int_0^L v_t^2\mathrm{d}z
\end{aligned}\tag{33}
$$

$$
\begin{aligned}
&-4L^2\Upsilon_3\rho_3\int_0^L w_z^2\mathrm{d}z-\frac{\rho_3}{\Upsilon_3}\int_0^L w_t^2\mathrm{d}z\leq\rho_3\int_0^L ww_t\mathrm{d}z\leq\\
&4L^2\Upsilon_3\rho_3\int_0^L w_z^2\mathrm{d}z+\frac{\rho_3}{\Upsilon_3}\int_0^L w_t^2\mathrm{d}z
\end{aligned}\tag{34}
$$

A calculation shows that

$$
\begin{aligned}
V\geq&\left(\frac{m_0}{2}-\frac{\rho_1}{\Upsilon_1}\right)\int_0^L u_t^2\mathrm{d}z+\left(\frac{m_0}{2}-\frac{\rho_2}{\Upsilon_2}\right)\int_0^L v_t^2\mathrm{d}z+\\
&\left(\frac{m_0}{2}-\frac{\rho_3}{\Upsilon_3}\right)\int_0^L w_t^2\mathrm{d}z+\left(\frac{P_0}{2}-4L^2\Upsilon_1\rho_1\right)\int_0^L u_z^2\mathrm{d}z+\\
&\left(\frac{P_0}{2}-4L^2\Upsilon_2\rho_2\right)\int_0^L v_z^2\mathrm{d}z+\left(\frac{EA}{2}-4L^2\Upsilon_3\rho_3\right)\int_0^L w_z^2\mathrm{d}z+\\
&\frac{EA}{4}\int_0^L\left(w_zu_z^2+w_zv_z^2\right)\mathrm{d}z+\frac{EI}{8}\int_0^L\left(u_z^4+v_z^4\right)\mathrm{d}z+\\
&\frac{EI}{2}\int_0^L\left(u_{zz}^2+v_{zz}^2\right)\mathrm{d}z+\frac{1}{2}\left(k_1+\frac{k_2\rho_1}{m_0}\right)u^2(L,t)+\\
&\frac{1}{2}\left(k_3+\frac{k_4\rho_2}{m_0}\right)v^2(L,t)+\left(k_5+\frac{k_6\rho_3}{m_0}\right)w^2(L,t)
\end{aligned}\tag{35}
$$

and

$$
\begin{aligned}
V\leq&\left(\frac{m_0}{2}+\frac{\rho_1}{\Upsilon_1}\right)\int_0^L u_t^2\mathrm{d}z+\left(\frac{m_0}{2}+\frac{\rho_2}{\Upsilon_2}\right)\int_0^L v_t^2\mathrm{d}z+\\
&\left(\frac{m_0}{2}+\frac{\rho_3}{\Upsilon_3}\right)\int_0^L w_t^2\mathrm{d}z+\left(\frac{P_0}{2}+4L^2\Upsilon_1\rho_1\right)\int_0^L u_z^2\mathrm{d}z+\\
&\left(\frac{P_0}{2}+4L^2\Upsilon_2\rho_2\right)\int_0^L v_z^2\mathrm{d}z+\left(\frac{EA}{2}+4L^2\Upsilon_3\rho_3\right)\int_0^L w_z^2\mathrm{d}z+\\
&\frac{EA}{4}\int_0^L\left(w_zu_z^2+w_zv_z^2\right)\mathrm{d}z+\frac{EI}{8}\int_0^L\left(u_z^4+v_z^4\right)\mathrm{d}z+\\
&\frac{EI}{2}\int_0^L\left(u_{zz}^2+v_{zz}^2\right)\mathrm{d}z+\frac{1}{2}\left(k_1+\frac{k_2\rho_1}{m_0}\right)u^2(L,t)+\\
&\frac{1}{2}\left(k_3+\frac{k_4\rho_2}{m_0}\right)v^2(L,t)+\left(k_5+\frac{k_6\rho_3}{m_0}\right)w^2(L,t)
\end{aligned}\tag{36}
$$

The selection of $\rho_1, \rho_2, \rho_3$, $\Upsilon_1$, $\Upsilon_2$, and $\Upsilon_3$ such that:

$$\begin{aligned}&\frac{m_0}{2}-\frac{\rho_1}{\Upsilon_1}=c_1, \frac{m_0}{2}-\frac{\rho_2}{\Upsilon_2}=c_2, \frac{m_0}{2}-\frac{\rho_3}{\Upsilon_3}=c_3 \\ &\frac{P_0}{2}-4L^2\Upsilon_1\rho_1=c_4, \frac{P_0}{2}-4L^2\Upsilon_2\rho_2=c_5, \frac{P_0}{2}-4L^2\Upsilon_3\rho_3=c_6\end{aligned} \tag{37}$$

where $c_i$, for $i$ = 1,…,6, are strictly positive constants. Eq. (37) shows that the Lyapunov candidate $V$ is a proper function of $\int_0^L u_t^2 \mathrm{d}z, \int_0^L v_t^2 \mathrm{d}z, \int_0^L w_t^2 \mathrm{d}z, \int_0^L u_z^2 \mathrm{d}z, \int_0^L v_z^2 \mathrm{d}z, \int_0^L w_z^2 \mathrm{d}z$, $\int_0^L u_{zz}^2 \mathrm{d}z$ and $\int_0^L v_{zz}^2 \mathrm{d}z$. Differentiating (25) along the solutions of the equations of motion (24) results in

$$\dot{V}=\Delta_1+\Delta_2+\Delta_3+\Delta_4+\Delta_5+\Delta_6+\Delta_7 \tag{38}$$

where

$$\begin{aligned}\Delta_1=\int_0^L u_t\bigg(&-EIu_{zzzz}+P_0u_{zz}+\frac{3EA}{2}u_z^2u_{zz}+EAw_{zz}u_z+ \\ &EAw_zu_{zz}+\frac{EA}{2}u_{zz}v_z^2+EAv_zv_{zz}u_z-\Omega_{1D}u_t+f_u\bigg)\mathrm{d}z\end{aligned} \tag{39}$$

$$\begin{aligned}\Delta_2=\int_0^L v_t\bigg(&-EIv_{zzzz}+P_0v_{zz}+\frac{3EA}{2}v_z^2v_{zz}+EAw_{zz}v_z+ \\ &EAw_zv_{zz}+\frac{EA}{2}v_{zz}u_z^2+EAu_zu_{zz}v_z-\Omega_{2D}v_t+f_v\bigg)\mathrm{d}z\end{aligned} \tag{40}$$

$$\Delta_3=\int_0^L w_t\left(EAw_{zz}+EAu_zu_{zz}+EAv_zv_{zz}-\Omega_{3D}w_t+f_w\right)\mathrm{d}z \tag{41}$$

$$\begin{aligned}&\Delta_4=P_0\int_0^L\left(u_zu_{zt}+v_zv_{zt}\right)\mathrm{d}z+EA\int_0^L w_zw_{zt}\mathrm{d}z+ \\ &EA\int_0^L\left(w_z+\frac{u_z^2}{2}+\frac{v_z^2}{2}\right)\left(w_{zt}+u_zu_{zt}+v_zv_{zt}\right)\mathrm{d}z+ \\ &EI\int_0^L\left(u_{zz}u_{zzt}+v_{zz}v_{zzt}\right)\mathrm{d}z+\left(k_1+\frac{k_2\rho_1}{m_0}\right)u(L,t)u_t(L,t)+ \\ &\left(k_3+\frac{k_4\rho_2}{m_0}\right)v(L,t)v_t(L,t)+\left(k_5+\frac{k_6\rho_3}{m_0}\right)w(L,t)w_t(L,t)\end{aligned} \tag{42}$$

$$\begin{aligned}\Delta_5=\rho_1\int_0^L u_t^2\mathrm{d}z+\frac{\rho_1}{m_0}\int_0^L u\big(&-EIu_{zzzz}+P_0u_{zz}+\frac{3EA}{2}u_z^2u_{zz}+ \\ &EAw_{zz}u_z+EAw_zu_{zz}+\frac{EA}{2}u_{zz}v_z^2+EAv_zv_{zz}u_z- \\ &\Omega_{1D}u_t+f_u\big)\mathrm{d}z\end{aligned} \tag{43}$$

$$\begin{aligned}\Delta_6=\rho_2\int_0^L v_t^2\mathrm{d}z+\frac{\rho_2}{m_0}\int_0^L v\big(&-EIv_{zzzz}+P_0v_{zz}+\frac{3EA}{2}v_z^2v_{zz}+ \\ &EAw_{zz}v_z+EAw_zv_{zz}+\frac{EA}{2}v_{zz}u_z^2+EAu_zu_{zz}v_z- \\ &\Omega_{2D}v_t+f_v\big)\mathrm{d}z\end{aligned} \tag{44}$$

$$\begin{aligned}\Delta_7=\rho_3\int_0^L w_t^2\mathrm{d}z+\frac{\rho_3}{m_0}\int_0^L w\big(&EAw_{zz}+EAu_zU_{zz}+EAv_zv_{zz}- \\ &\Omega_{3D}w_t+f_w\big)\mathrm{d}z\end{aligned} \tag{45}$$

Integrating (39), (40), (41), (43), (44), and (45) by parts then substituting the result into (38) and using boundary conditions give

$$\begin{aligned}&\dot{V}=\left(u_t(L,t)+\frac{\rho_1}{m_0}u(L,t)\right)\Big(-EIu_{zzz}(L,t)+P_0u_z(L,t)+ \\ &\frac{EA}{2}u_z^3(L,t)+EAw_z(L,t)u_z(L,t)+ \\ &\frac{EA}{2}u_z(L,t)v_z^2(L,t)\Big)+\left(v_t(L,t)+\frac{\rho_2}{m_0}v(L,t)\right) \\ &\Big(-EIv_{zzz}(L,t)+P_0v_z(L,t)+\frac{EA}{2}v_z^3(L,t)+ \\ &EAw_z(L,t)v_z(L,t)+\frac{EA}{2}v_z(L,t)v_z^2(L,t)\Big)+ \\ &\left(w_t(L,t)+\frac{\rho_3}{m_0}w(L,t)\right)\Big(EAw_z(L,t)+\frac{EA}{2}u_z^2(L,t)+ \\ &\frac{EA}{2}v_z^2(L,t)\Big)-\left(\Omega_{1D}-\rho_1\right)\int_0^L u_t^2\mathrm{d}z-\left(\Omega_{2D}-\rho_2\right)\int_0^L v_t^2\mathrm{d}z- \\ &\left(\Omega_{3D}-\rho_3\right)\int_0^L w_t^2\mathrm{d}z-\frac{\rho_1EI}{m_0}\int_0^L u_{zz}^2\mathrm{d}z-\frac{\rho_2EI}{m_0}\int_0^L v_{zz}^2\mathrm{d}z- \\ &\frac{\rho_1P_0}{m_0}\int_0^L u_z^2\mathrm{d}z-\frac{\rho_2P_0}{m_0}\int_0^L v_z^2\mathrm{d}z-\frac{\rho_1EA}{2m_0}\int_0^L u_z^4\mathrm{d}z- \\ &\frac{\rho_2EA}{2m_0}\int_0^L v_z^4\mathrm{d}z-\frac{EA}{m_0}(\rho_1+\frac{\rho_3}{2})\int_0^L u_z^2w_z\mathrm{d}z- \\ &\frac{EA}{2m_0}(\rho_1+\rho_2)\int_0^L u_z^2v_z^2\mathrm{d}z-\frac{EA}{m_0}(\rho_2+\frac{\rho_3}{2})\int_0^L v_z^2w_z\mathrm{d}z- \\ &\frac{\rho_1\Omega_{1D}}{m_0}\int_0^L uu_t\mathrm{d}z+\frac{\rho_1}{m_0}\int_0^L uf_u\mathrm{d}z-\frac{\rho_2\Omega_{2D}}{m_0}\int_0^L vv_t\mathrm{d}z+ \\ &\frac{\rho_2}{m_0}\int_0^L vf_v\mathrm{d}z-\frac{\rho_3EA}{m_0}\int_0^L w_z^2\mathrm{d}z-\frac{\rho_3\Omega_{3D}}{m_0}\int_0^L ww_t\mathrm{d}z+ \\ &\int_0^L v_tf_v\mathrm{d}z+\int_0^L u_tf_u\mathrm{d}z+\int_0^L w_tf_w\mathrm{d}z+\frac{\rho_3}{m_0}\int_0^L wf_w\mathrm{d}z+ \\ &(k_1+\frac{k_2\rho_1}{m_0})u(L,t)u_t(L,t)+(k_3+\frac{k_4\rho_2}{m_0})v(L,t)v_t(L,t)+ \\ &(k_5+\frac{k_6\rho_3}{m_0})w(L,t)w_t(L,t)\end{aligned} \tag{46}$$

See Appendix B for detailed integrations by parts of (39), (40), (41), (43), (44), and (45). Since

$$-\frac{\Omega_{1D}\rho_1}{m_0}\int_0^L uu_t\mathrm{d}z\le\frac{4L^2\Omega_{1D}\rho_1\Upsilon_4}{m_0}\int_0^L u_z^2\mathrm{d}z+\frac{\Omega_{1D}\rho_1}{\Upsilon_4m_0}\int_0^L u_t^2\mathrm{d}z \tag{47}$$

$$-\frac{\Omega_{2D}\rho_2}{m_0}\int_0^L vv_t\mathrm{d}z\le\frac{4L^2\Omega_{2D}\rho_2\Upsilon_5}{m_0}\int_0^L v_z^2\mathrm{d}z+\frac{\Omega_{2D}\rho_2}{\Upsilon_5m_0}\int_0^L v_t^2\mathrm{d}z \tag{48}$$

$$-\frac{\Omega_{3D}\rho_3}{m_0}\int_0^L ww_t\mathrm{d}z\le\frac{4L^2\Omega_{3D}\rho_3\Upsilon_6}{m_0}\int_0^L w_z^2\mathrm{d}z+\frac{\Omega_{3D}\rho_3}{\Upsilon_6m_0}\int_0^L w_t^2\mathrm{d}z \tag{49}$$

and recall that

$$\begin{aligned}&-EIu_{zzz}(L,t)+P_0u_z(L,t)+\frac{EA}{2}u_z^3(L,t)+ \\ &EAw_z(L,t)u_z(L,t)+\frac{EA}{2}u_z(L,t)v_z^2(L,t)=U_u\end{aligned}$$

$$-EIv_{zz}(L,t)+P_0v_z(L,t)+\frac{EA}{2}v_z^3(L,t)+$$
$$EAw_z(L,t)v_z(L,t)+\frac{EA}{2}v_z(L,t)u_z^2(L,t)=U_v$$

and $EAw_z(L,t)+\frac{EA}{2}u_z^2(L,t)+\frac{EA}{2}v_z^2(L,t)=U_w$, the boundary control can be selected as follows,

$$U_u=-k_1u(L,t)-k_2u_t(L,t) \tag{50}$$

$$U_v=-k_3v(L,t)-k_4v_t(L,t) \tag{51}$$

$$U_w=-k_5w(L,t)-k_6w_t(L,t) \tag{52}$$

where coefficients $k_i$, for $i$=1,...,6, are strictly positive constants. Substituting the controls (50), (51), and (52) into (46) gives

$$\begin{aligned}\dot V\le&-\frac{k_1\rho_1}{m_0}u^2(L,t)-k_2u_t^2(L,t)-\frac{k_3\rho_2}{m_0}v^2(L,t)-\\&k_4v_t^2(L,t)-\frac{k_5\rho_3}{m_0}w^2(L,t)-k_6w_t^2(L,t)-(\Omega_{1D}-\\&\rho_1-\frac{\Omega_{1D}\rho_1}{\Upsilon_4m_0})\int_0^Lu_t^2\mathrm{d}z-(\Omega_{2D}-\rho_2-\frac{\Omega_{2D}\rho_2}{\Upsilon_5m_0})\int_0^Lv_t^2\mathrm{d}z-\\&(\Omega_{3D}-\rho_3-\frac{\Omega_{3D}\rho_3}{\Upsilon_6m_0})\int_0^Lw_t^2\mathrm{d}z-\frac{\rho_1EI}{m_0}\int_0^Lu_{zz}^2\mathrm{d}z-\\&\frac{\rho_2EI}{m_0}\int_0^Lv_{zz}^2\mathrm{d}z-(\frac{\rho_1P_0}{m_0}-\frac{4L^2\Omega_{1D}\rho_1\Upsilon_4}{m_0})\int_0^Lu_z^2\mathrm{d}z-\\&(\frac{\rho_2P_0}{m_0}-\frac{4L^2\Omega_{2D}\rho_2\Upsilon_5}{m_0})\int_0^Lv_z^2\mathrm{d}z-(\frac{\rho_3EA}{m_0}-\\&\frac{4L^2\Omega_{3D}\rho_3\Upsilon_6}{m_0})\int_0^Lw_z^2\mathrm{d}z-\frac{\rho_1EA}{2m_0}\int_0^Lu_z^4\mathrm{d}z-\\&\frac{\rho_2EA}{2m_0}\int_0^Lv_z^4\mathrm{d}z-\frac{EA}{m_0}(\rho_1+\frac{\rho_3}{2})\int_0^Lu_z^2\mathrm{w}_z\mathrm{d}z-\\&\frac{EA}{2m_0}(\rho_1+\rho_2)\int_0^Lu_z^2v_z^2\mathrm{d}z-\frac{EA}{m_0}(\rho_2+\frac{\rho_3}{2})\int_0^Lv_z^2w_z\mathrm{d}z-\\&\frac{\rho_1\Omega_{1D}}{m_0}\int_0^Luu_t\mathrm{d}z+\frac{\rho_1}{m_0}\int_0^Luf_u\mathrm{d}z-\frac{\rho_2\Omega_{2D}}{m_0}\int_0^Lvv_t\mathrm{d}z+\\&\frac{\rho_2}{m_0}\int_0^Lvf_v\mathrm{d}z-\frac{\rho_3\Omega_{3D}}{m_0}\int_0^Lww_t\mathrm{d}z+\int_0^Lv_tf_v\mathrm{d}z+\\&\int_0^Lu_tf_u\mathrm{d}z+\int_0^Lw_tf_w\mathrm{d}z+\frac{\rho_3}{m_0}\int_0^Lwf_w\mathrm{d}z\end{aligned} \tag{53}$$

From (53), the designed parameters are selected such that

$$\begin{aligned}&\Omega_{1D}-\rho_1-\frac{\Omega_{1D}\rho_1}{\Upsilon_4m_0}=c_7,\quad \Omega_{2D}-\rho_2-\frac{\Omega_{2D}\rho_2}{\Upsilon_5m_0}=c_8\\&\Omega_{3D}-\rho_3-\frac{\Omega_{3D}\rho_3}{\Upsilon_5m_0}=c_9,\quad \frac{\rho_1P_0}{m_0}-\frac{4L^2\Omega_{1D}\rho_1\Upsilon_4}{m_0}=c_{10}\\&\frac{\rho_2P_0}{m_0}-\frac{4L^2\Omega_{2D}\rho_2\Upsilon_5}{m_0}=c_{11},\quad \frac{\rho_3P_0}{m_0}-\frac{4L^2\Omega_{3D}\rho_3\Upsilon_6}{m_0}=c_{12}\end{aligned} \tag{54}$$

where $c_i$, for $i$=7,…,12, are strictly positive constants. Using the upper bound of $V$ specified in (36), (53) can be expressed as

$$\begin{aligned}\dot V\le&-\frac{k_1\rho_1}{m_0}u^2(L,t)-k_2u_t^2(L,t)-\frac{k_3\rho_2}{m_0}v^2(L,t)-\\&k_4v_t^2(L,t)-\frac{k_5\rho_3}{m_0}w^2(L,t)-k_6w_t^2(L,t)-cV+\\&\frac{\rho_1}{m_0}\int_0^Luf_u\mathrm{d}z+\frac{\rho_2}{m_0}\int_0^Lvf_v\mathrm{d}z+\int_0^Lv_tf_v\mathrm{d}z+\int_0^Lu_tf_u\mathrm{d}z+\\&\int_0^Lw_tf_w\mathrm{d}z+\frac{\rho_3}{m_0}\int_0^Lwf_w\mathrm{d}z\end{aligned} \tag{55}$$

where

$$\begin{aligned}&c=\min\{c_7,c_8,c_9,c_{10},c_{11},c_{12},\frac{\rho_1EI}{m_0},\frac{\rho_2EI}{m_0},\\&\frac{\rho_1EA}{m_0},\frac{\rho_2EI}{m_0},\frac{EA}{2m_0}(\rho_1+\rho_2),\beta_1\}/\\&\max\{\frac{m_0}{2}+\frac{\rho_1}{\Upsilon_1},\frac{m_0}{2}+\frac{\rho_2}{\Upsilon_2},\frac{m_0}{2}+\frac{\rho_3}{\Upsilon_3},\frac{P_0}{2}+4L^2\Upsilon_1\rho_1,\\&\frac{P_0}{2}+4L^2\Upsilon_2\rho_2,\frac{EA}{2}+4L^2\Upsilon_3\rho_3,\frac{EA}{2},\frac{EI}{2},\beta_2\}\end{aligned} \tag{56}$$

where

$$\begin{aligned}&\beta_1=\left\{\frac{EA}{m_0}\left(\rho_1+\frac{\rho_3}{2}\right),\frac{EA}{m_0}\left(\rho_2+\frac{\rho_3}{2}\right),k_1\frac{\rho_1}{m_0},k_3\frac{\rho_2}{m_0},k_5\frac{\rho_3}{m_0}\right\}\\&\beta_2=\left\{\frac{1}{2}\left(k_1+\frac{k_2\rho_1}{m_0}\right),\frac{1}{2}\left(k_3+\frac{k_4\rho_2}{m_0}\right),\frac{1}{2}\left(k_5+\frac{k_6\rho_3}{m_0}\right)\right\}\end{aligned} \tag{57}$$

At this point, the main outcome of this paper is stated in the following theorem.

**Theorem** Under Assumption 1, the control inputs $U_u$, $U_v$, and $U_w$ given in (50), (51), and (52) solve the control objective provided that the design constants $\rho_1$, $\rho_2$, and $\rho_3$ are chosen such that the conditions specified in (54) hold. In particular, the solutions of the closed-loop system consisting of (24), (50), (51), and (52) exist and are unique. Moreover, when the external distributed disturbances $f_u$, $f_v$, and $f_w$ are zero, all the terms $|u(z,t)|,|v(z,t)|,|w(z,t)|,\int_0^Lu_z(z,t)\mathrm{d}z,$ $\int_0^Lv_z(z,t)\mathrm{d}s,\int_0^Lw_z(z,t)\mathrm{d}s,\int_0^Lu_t(z,t)\mathrm{d}z,\int_0^Lv_t(z,t)\mathrm{d}z,\int_0^Lw_t(z,t)\mathrm{d}z,$ $\int_0^Lu_{zz}\mathrm{d}z,$ and $\int_0^Lv_{zz}\mathrm{d}z$ exponentially converge to zero $\forall z\in[0,L]$ and $\forall z\ge t_0$, and when the external distributed disturbances $f_u$, $f_v$, and $f_w$ are different from zero but bounded, all the terms $|u(z,t)|,|v(z,t)|,|w(z,t)|,\int_0^Lu_z(z,t)\mathrm{d}z,$ $\int_0^Lv_z(z,t)\mathrm{d}s,\int_0^Lw_z(z,t)\mathrm{d}s,\int_0^Lu_t(z,t)\mathrm{d}z,\int_0^Lv_t(z,t)\mathrm{d}z,\int_0^Lw_t(z,t)\mathrm{d}z,$ $\int_0^Lu_{zz}\mathrm{d}z,$ and $\int_0^Lv_{zz}\mathrm{d}z$ exponentially converge to some small positive constants $\forall z\in[0,L]$ and $\forall z\ge t_0$.

**Table 1 The marine riser system parameters**

| Nomenclature | Value |
|---|---|
| Length $L$/m | 1 000 |
| Diameter $D$/m | 0.2 |
| Density$\rho$/(kg·m$^{-3}$) | 8200 |
| Young's modulus $E$/(kg·m$^{-2}$) | $2\times10^{8}$ |
| Tension $P_0$/kN | $1.1\times10^{4}$ |
| Drag velocity coefficient $C_D$ | 1.2 |
| Drag acceleration coefficient $C_M$ | 1.4 |
| Lift force coefficient $C_L$ | 0.6 |
| $X$-direction damping$\Omega_{1D}$/(s·m$^{-2}$) | 80 |
| $Y$-direction damping$\Omega_{2D}$/(s·m$^{-2}$) | 80 |
| $Z$-direction damping$\Omega_{3D}$/(s·m$^{-2}$) | 60 |

## 4 Numerical simulations

The effectiveness of the proposed control is illustrated by several numerical simulations. The parameters of the marine riser system taken from Do and Pan (2008) are given in Table 1.

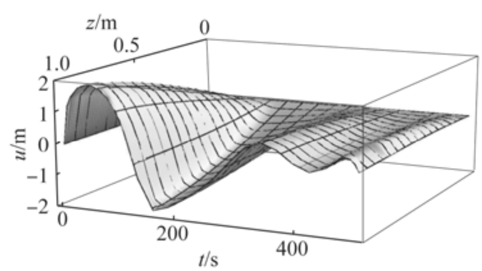
The water particle velocities in the $X$ and $Y$ directions $u_1(z, t)$ and $u_2(z, t)$ are expressed as Niedzwecki and Liagre (2003)

$$u_{1,2}\left(z,t\right)=\sum_{i=1}^{N_w}\left(A_{wi}w_{wi}\frac{\cosh\left(k_{wi}z\right)}{\sinh\left(k_{wi}L\right)}\sin\left(w_{wi}t+\phi_{wi}\right)\right) \tag{58}$$

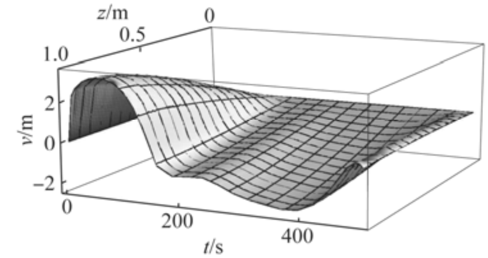
where the amplitude $A_{wi}$, wave number $k_{wi}$, frequency $w_{wi}$, phase $\phi_{wi}$ of the wave $i$th are given by

$$\begin{aligned}&w_{wi}=w_m+\frac{w_m+w_M}{N_w}i, S_{wi}=\frac{1.25}{4}\frac{w_0^4}{w_{sw}^5}H_{sw}^2\mathrm{e}^{-1.25\left(w_0/w_{wi}\right)^4}\\&A_{wi}=\sqrt{2S_{wi}\frac{w_{mi}-w_{Mi}}{n_w}}, 9.8k_{wi}\tanh\left(k_{wi}L\right)=w_{wi}^2, \phi_{wi}=2\pi\mathrm{rand}()\end{aligned} \tag{59}$$

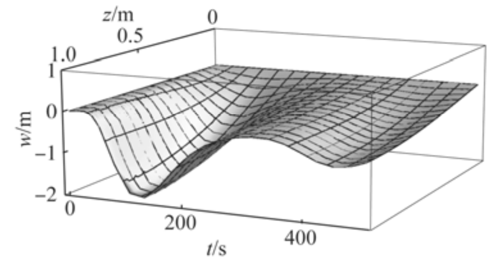
In (59), minimum and maximum wave frequencies are $w_m$=0.1rand/s and $w_M$=1.5rand/s; the significant wave height $H_{sw}$=4m; the modal frequency is $w_0=2\pi/T_w$ with the period $T_w$=7.8; $N_w$=5; and rand() is a random number between 0 and 1. The control gains are selected to be $k_1=k_2=k_3=k_4=190$. A simple verification shows that the selected control gains simultaneously satisfy the conditions given in (37) and (54) with $\rho_1$, $\rho_2$, $\rho_3$, $\Upsilon_1$, $\Upsilon_2$, $\Upsilon_3$, $\Upsilon_4$, $\Upsilon_5$, $\Upsilon_6$, $\Upsilon_7=10^{-3}$. The initial conditions at $t_0 = 0$ are $u(z, t_0)=v(z, t_0)=0$ and $u_t(z, t_0)=v_t(z, t_0)=0$. In this paper, finite difference method is used for numerical simulation purposes. Simulations are carried out over 500 seconds without the proposed control ($k_1=k_2=k_3=k_4=0$) and with the proposed control.




(a) $u(z, t)$

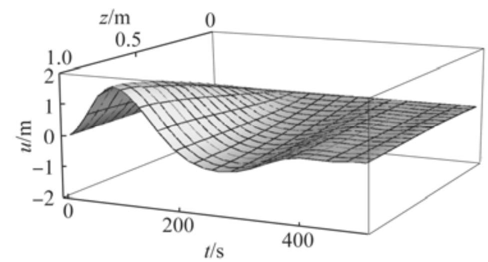



(b) $v(z, t)$

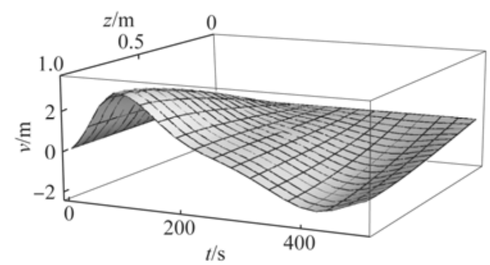



(c) $w(z, t)$

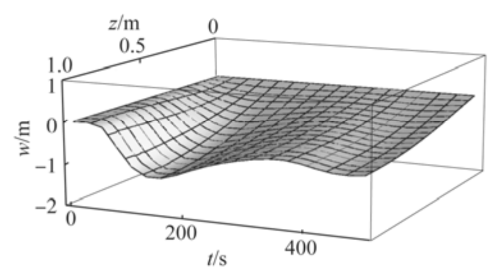
**Fig. 2 The riser's displacements without control**




(a) $u(z, t)$



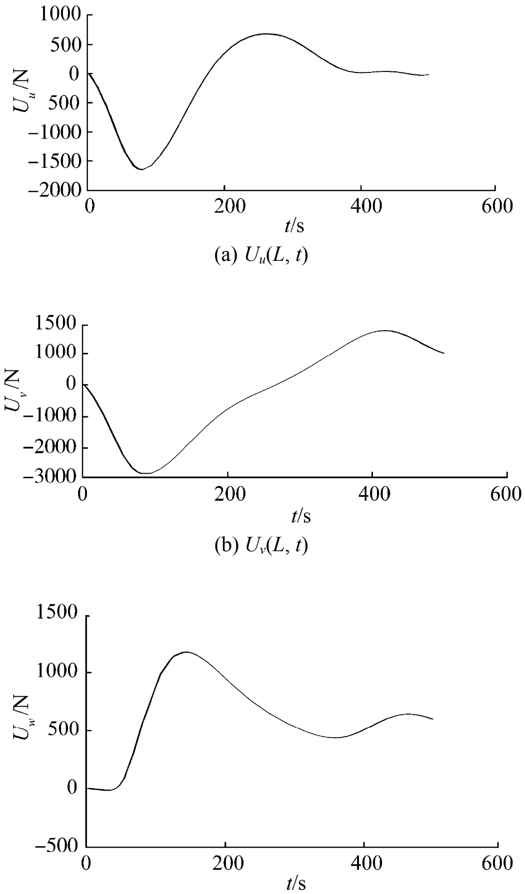

(b) $v(z, t)$




(c) $w(z, t)$

**Fig. 3 The riser's displacements with control**

The transverse displacements in uncontrolled and controlled cases are plotted in Figs. 2, 3, respectively. Fig. 3 shows that transverse and longitudinal displacements are reduced significantly when the proposed control is applied. The proposed control is able to drive the riser to the vicinity of its equilibrium position. Fig. 4 indicates that control forces are in an implementable range in practice.



(a) $U_u(L, t)$

(b) $U_v(L, t)$

(c) $U_w(L, t)$

**Fig. 4 Control forces**

## 5 Conclusions

The equations of motion which indicate strong nonlinear couplings for a marine riser system in three dimensions were derived using the extended Hamilton's principle. Subsequently, the Lyapunov direct method was employed to design the boundary controller applied at the top end of the riser. The designed controller's ability to stabilize the riser at its equilibrium position was proven analytically and illustrated numerically. The main contributions of this paper were the introduction of the Lyapunov function candidate (25) and the riser model in three dimensions with nonlinear couplings (24). An extension of this work is to include torsion to the riser dynamics.

## Appendix A: Useful inequalities

Two useful inequalities (Do and Pan, 2008) that will be used extensively in proving Theorem 1. are provided.

**Inequality A.1** For any $y = [y_1,\dots, y_i,\dots, y_n]^{\mathrm{T}}$ with $y_i \in C^1[0,L]$, for $i$=1,…,$n$, the following inequalities hold:

$$\int_0^L y(s)\cdot y(s)\mathrm{d}s \le 2Ly(0)\cdot y(0)+4L^2\int_0^L y_s(s)\cdot y_s(s)\mathrm{d}s \tag{A1}$$

$$\begin{aligned}\int_0^L y(s)\cdot y(s)\mathrm{d}s \le\ & 2Ly(L,t)\cdot y(L,t)+\\ & 4L^2\int_0^L y_s(s)\cdot y_s(s)\mathrm{d}s\end{aligned} \tag{A2}$$

**Inequality A.2** For any $y = [y_1,\dots, y_i,\dots, y_n]^{\mathrm{T}}$ with $y_i \in C^1[0,L]$, $i$=1, …, $n$, the following inequalities hold:

$$\begin{aligned}\max_{s\in[0,L]}\big(y(s)\cdot y(s)\big) \le\ & y(0)\cdot y(0)+\\ & 2\sqrt{\int_0^L y(s)\cdot y(s)\mathrm{d}s}\sqrt{\int_0^L y_s(s)\cdot y_s(s)\mathrm{d}s}\end{aligned} \tag{A3}$$

## Appendix B: Integrations by parts results

Integrations by parts of $\Delta_1$, $\Delta_3$, $\Delta_4$, $\Delta_5$, $\Delta_6$, and $\Delta_7$ are provided in the appendix.

$$\begin{aligned}\Delta_1 =\ & \int_0^L u_t\Big(-EIu_{zzzz}+P_0u_{zz}+\frac{3EA}{2}u_z^2u_{zz}+EAw_{zz}u_z+\\ & EAw_zu_{zz}+\frac{EA}{2}u_{zz}v_z^2+EAv_zv_{zz}u_z-\varOmega_{1D}u_t+f_u\Big)\mathrm{d}z=\\ & -EIu_tu_{zzz}\Big|_0^L+EIu_{zt}u_{zz}\Big|_0^L-EI\int_0^L u_{zzt}u_{zz}\mathrm{d}z+\\ & P_0u_tu_z\Big|_0^L-P_0\int_0^L u_{zt}u_z\mathrm{d}z+\frac{EA}{2}u_tu_z^3\Big|_0^L-\\ & \frac{EA}{2}\int_0^L u_z^3u_{zt}\mathrm{d}z+\frac{EA}{2}u_tv_z^2u_z\Big|_0^L-\\ & \frac{EA}{2}\int_0^L u_zu_{zt}v_z^2\mathrm{d}z+EAu_tw_zu_z\Big|_0^L-\\ & EA\int_0^L w_zu_{zt}u_z\mathrm{d}z-\varOmega_{1D}\int_0^L u_t^2\mathrm{d}z+\int_0^L u_tf_u\mathrm{d}z\end{aligned} \tag{B1}$$

$$\begin{aligned}\Delta_2 =\ & \int_0^L v_t\Big(-EIv_{zzzz}+P_0v_{zz}+\frac{3EA}{2}v_z^2v_{zz}+\\ & EAw_{zz}v_z+EAw_zv_{zz}+\frac{EA}{2}v_{zz}u_z^2+EAu_zu_{zz}v_z-\\ & \varOmega_{2D}v_t+f_v\Big)\mathrm{d}z=-EIv_tv_{zzz}\Big|_0^L+EIv_{zt}v_{zz}\Big|_0^L-\\ & EI\int_0^L v_{zzt}v_{zz}\mathrm{d}z+P_0v_tv_z\Big|_0^L-P_0\int_0^L v_{zt}v_z\mathrm{d}z+\\ & \frac{EA}{2}v_tv_z^3\Big|_0^L-\frac{EA}{2}\int_0^L v_z^3v_{zt}\mathrm{d}z+\\ & \frac{EA}{2}v_tu_z^2v_z\Big|_0^L-\frac{EA}{2}\int_0^L v_zv_{zt}u_z^2\mathrm{d}z+EAv_tw_zv_z\Big|_0^L-\\ & EA\int_0^L w_zv_{zt}v_z\mathrm{d}z-\varOmega_{2D}\int_0^L v_t^2\mathrm{d}z+\int_0^L v_tf_v\mathrm{d}z\end{aligned} \tag{B2}$$

$$\begin{aligned}\Delta_3 =\ & \int_0^L w_t(EAw_{zz}+EAu_zu_{zz}+EAv_zv_{zz}-\varOmega_{3D}w_t+f_w)\mathrm{d}z=\\ & EAw_tw_z\Big|_0^L-EA\int_0^L w_{zt}w_z\mathrm{d}z+\frac{EA}{2}w_tu_z^2\Big|_0^L-\\ & \frac{EA}{2}\int_0^L w_{zt}u_z^2\mathrm{d}z+\frac{EA}{2}w_tv_z^2\Big|_0^L-\frac{EA}{2}\int_0^L w_{zt}v_z^2\mathrm{d}z-\\ & \varOmega_{3D}\int_0^L w_t^2\mathrm{d}z+\int_0^L w_tf_w\mathrm{d}z\end{aligned} \tag{B3}$$

$$
\begin{aligned}
\Delta_5 = {} & \rho_1\int_0^L u_t^2 \mathrm{d}z + \frac{\rho_1}{m_0}\int_0^L u(-EIu_{zzzz} + P_0 u_{zz} + \frac{3EA}{2}u_z^2 u_{zz} + \\
& EAw_{zz}u_z + EAw_z u_{zz} + \frac{EA}{2}u_{zz}v_z^2 + EAv_z v_{zz} u_z - \\
& \Omega_D u_t + f_u)\mathrm{d}z = \\
& \rho_1\int_0^L u_t^2\mathrm{d}z - \frac{\rho_1 EI}{m_0}uu_{zzz}\Big|_0^L + \frac{\rho_1 EI}{m_0}u_z u_{zz}\Big|_0^L - \\
& \frac{\rho_1 EI}{m_0}\int_0^L u_{zz}^2\mathrm{d}z + \frac{\rho_1 P_0}{m_0}uu_z\Big|_0^L - \frac{\rho_1 P_0}{m_0}\int_0^L u_z^2\mathrm{d}z + \\
& \frac{\rho_1 EA}{m_0}uu_z^3\Big|_0^L - \frac{\rho_1 A}{m_0}\int_0^L u_z^4\mathrm{d}z + \frac{\rho_1 EA}{m_0}uw_z u_z\Big|_0^L - \\
& \frac{\rho_1 EA}{m_0}\int_0^L u_z^2 w_z\mathrm{d}z + \frac{\rho_1 EA}{2m_0}uv_z^2 u_z\Big|_0^L - \frac{\rho_1 EA}{2m_0}\int_0^L u_z^2 v_z^2\mathrm{d}z - \\
& \frac{\rho_1 \Omega_D}{m_0}\int_0^L uu_t\mathrm{d}z + \frac{\rho_1}{m_0}\int_0^L f_u u\mathrm{d}z
\end{aligned} \tag{B4}
$$

$$
\begin{aligned}
\Delta_6 = {} & \rho_2\int_0^L v_t^2 \mathrm{d}z + \frac{\rho_2}{m_0}\int_0^L v(-EIv_{zzzz} + P_0 v_{zz} + \frac{3EA}{2}v_z^2 v_{zz} + \\
& EAw_{zz}v_z + EAw_z v_{zz} + \frac{EA}{2}v_{zz}u_z^2 + EAu_z u_{zz} v_z - \\
& \Omega_{2D} v_t + fv)\mathrm{d}z = \\
& \rho_2\int_0^L v_t^2\mathrm{d}z - \frac{\rho_v EI}{m_0}vv_{zzz}\Big|_0^L + \frac{\rho_2 EI}{m_0}v_z v_{zz}\Big|_0^L - \\
& \frac{\rho_1 EI}{m_0}\int_0^L v_{zz}^2\mathrm{d}z + \frac{\rho_2 P_0}{m_0}vv_z\Big|_0^L - \frac{\rho_2 P_0}{m_0}\int_0^L v_z^2\mathrm{d}z + \\
& \frac{\rho_2 EA}{m_0}vv_z^3\Big|_0^L - \frac{\rho_2 A}{m_0}\int_0^L v_z^4\mathrm{d}z + \frac{\rho_2 EA}{m_0}vw_z v_z\Big|_0^L - \\
& \frac{\rho_2 EA}{m_0}\int_0^L v_z^2 w_z\mathrm{d}z + \frac{\rho_2 EA}{2m_0}vu_z^2 v_z\Big|_0^L - \frac{\rho_1 EA}{2m_0}\int_0^L v_z^2 u_z^2\mathrm{d}z - \\
& \frac{\rho_2 \Omega_{1D}}{m_0}\int_0^L vv_t\mathrm{d}z + \frac{\rho_2}{m_0}\int_0^L f_v v\mathrm{d}z
\end{aligned} \tag{B5}
$$

$$
\begin{aligned}
\Delta_7 = {} & \rho_3\int_0^L w_t^2 \mathrm{d}z + \frac{\rho_3}{m_0}\int_0^L w(EAw_{zz} + EAu_z u_{zz} + EAv_z v_{zz} - \\
& \Omega_{3D} w_t + f_w)\mathrm{d}z = \\
& \rho_3\int_0^L w_t^2\mathrm{d}z + \frac{\rho_3 EA}{m_0}ww_z\Big|_0^L - \frac{\rho_3 EA}{m_0}\int_0^L w_z^2\mathrm{d}z + \\
& \frac{\rho_3 EA}{2m_0}wu_z^2\Big|_z^2 - \frac{\rho_3 EA}{2m_0}\int_0^L w_z u_z^2\mathrm{d}z + \frac{\rho_3 EA}{2m_0}wv_z^2\Big|_z^2 - \\
& \frac{\rho_3 EA}{2m_0}\int_0^L w_z v_z^2\mathrm{d}z - \frac{\rho_3 \Omega_{3D}}{m_0}\int_0^L ww_t\,\mathrm{d}z + \frac{\rho_3}{m_0}\int_0^L wf_w\mathrm{d}z
\end{aligned} \tag{B6}
$$

## Appendix C: Proof of Theorem 1

Proof of the existence and uniqueness of the solutions to the closed-loop system consisting of (24), (50), and (51), and then a proof of convergence of the solutions is given. Proof of the existence and uniqueness follows the same procedure as in Do and Pan (2008).

### C.1 Proof of existence

Define $H^2(0,L)$ as the usual Hilbert space. The process of proving existence and uniqueness is based on the Sobolev spaces:

$$V_S = \theta \in H^2\left(0,L\right)\Big|_{\theta(0,t)=0} \tag{C1}$$

equipped with the norm $\|\bullet\|_{V_S} = \|\theta_{ss}\|_2$, and

$$W_S = \theta \in V_S \cap H^4\left(0,L\right)\Big|_{\theta_{ss}(0,t)=0,\theta_{ss}(L,t)=0} \tag{C2}$$

equipped with the norm $\|\bullet\|_{W_S} = \|\theta_{ss}\|_2 + \|\theta_{ssss}\|_2$, where $\|\bullet\|_p$ represents the $L^p$ norms. Taking inner products of both sides of the first three equations of (24) by $\Phi_1$, $\Phi_2$, $\Phi_3 \in V_S$, respectively, and integrating from 0 to $L$ result in

$$
\begin{aligned}
& -m_0\int_0^L u_{tt}\phi_1\mathrm{d}z - EI\int_0^L u_{zz}\phi_{1zz}\mathrm{d}z - P_0\int_0^L u_z\phi_{1z}\mathrm{d}z - \\
& \frac{EA}{2}\int_0^L u_z^3\phi_{1z}\mathrm{d}z - \Omega_{1D}\int_0^L u_t\phi_1\mathrm{d}z - \frac{EA}{2}\int_0^L u_z v_z^2\phi_{1z}\mathrm{d}z - \\
& EA\int_0^L w_z u_z\phi_{1z}\mathrm{d}z + \int_0^L f_u\phi_1\mathrm{d}z - [k_1 u(L,t) + k_2 u_t(L,t)] \\
& \phi_1(L,t) = 0
\end{aligned} \tag{C3}
$$

$$
\begin{aligned}
& -m_0\int_0^L v_{tt}\phi_2\mathrm{d}z - EI\int_0^L v_{zz}\phi_{2zz}\mathrm{d}z - P_0\int_0^L v_z\phi_{2z}\mathrm{d}z - \\
& \frac{EA}{2}\int_0^L v_z^3\phi_{2z}\mathrm{d}z - \Omega_{D}\int_0^L v_t\phi_2\mathrm{d}z - \frac{EA}{2}\int_0^L v_z u_z^2\phi_{2z}\mathrm{d}z - \\
& EA\int_0^L w_z v_z\phi_{2z}\mathrm{d}z + \int_0^L f_v\phi_2\mathrm{d}z - [k_3 v(L,t) + k_4 v_t(L,t)] \\
& \phi_2(L,t) = 0
\end{aligned} \tag{C4}
$$

$$
\begin{aligned}
& -m_0\int_0^L w_{tt}\phi_3\mathrm{d}z + EA\int_0^L w_z\phi_{3z}\mathrm{d}z - \frac{EA}{2}\int_0^L u_z^2\phi_{3z}\mathrm{d}z - \\
& \frac{EA}{2}\int_0^L v_z^2\phi_{3z}\mathrm{d}z - \Omega_{3D}\int_0^L w_t\phi_3\mathrm{d}z + \int_0^L f_w\phi_3\mathrm{d}z - [k_5 w(L,t) + \\
& k_6 w_t(L,t)]\phi_3(L,t) = 0
\end{aligned} \tag{C5}
$$

Galerkin's approximation is used to show that for all $\Phi_1$, $\Phi_2$, $\Phi_3 \in V_S$ there exist $u$, $v \in W_S$ and $w \in V_S$ such that (C3), (C4), and (C5) hold. Define $\phi_{1,2}^j$ and $\phi_3^i$ as components of complete orthogonal systems of $W_S$ and $V_S$, respectively, for which $\{u(z,t_0),u_t(z,t_0)\} \in$ Span $\{\phi_1^1,\phi_1^2\}$, $\{v(z,t_0),v_t(z,t_0)\} \in$ Span $\{\phi_2^1,\phi_2^2\}$, and $\{w(z,t_0),w_t(z,t_0)\} \in$ Span $\{\phi_3^1,\phi_3^2\}$. For each $n \in N$, let $W_{S1n}$=Span $\{\phi_1^1, \phi_1^2,\ldots,\phi_1^n\}$, $W_{S2n}$=Span $\{\phi_2^1,\phi_2^2,\ldots,\phi_2^n\}$, $V_{Sn}$=Span $\{\phi_3^1,\phi_3^2, \ldots,\phi_3^n\}$, we search for functions $u^n(s,t) = \sum_{j=1}^n k_1^j(t)\phi_1^j$, $v^n(s,t) = \sum_{j=1}^n k_2^j(t)\phi_2^j$, and $w^n(s,t) = \sum_{i=1}^n k_3^j(t)\phi_2^j$ that satisfy the following equations for all $\Phi_1 \in W_{S1n}$, $\Phi_2 \in W_{S2n}$, and $\Phi_3 \in W_{Sn}$

$$
\begin{aligned}
& -m_0\int_0^L u_{tt}^n\phi_1^n\mathrm{d}z - EI\int_0^L u_{zz}^n\phi_{1zz}^n\mathrm{d}z - P_0\int_0^L u_z^n\phi_{1z}^n\mathrm{d}z - \\
& \frac{EA}{2}\int_0^L u_z^{3n}\phi_{1z}^n\mathrm{d}z - \Omega_{1D}\int_0^L u_t^n\phi_1^n\mathrm{d}z - \frac{EA}{2}\int_0^L u_z^n v_z^{2n}\phi_{1z}^n\mathrm{d}z - \\
& EA\int_0^L w_z^n u_z^n\phi_{2z}^n\mathrm{d}z + \int_0^L f_u^n u^n\mathrm{d}z - [k_1 u^n\left(L,t\right) + k_2 u_t^n\left(L,t\right)] \\
& \phi_1^n\left(L,t\right) = 0
\end{aligned} \tag{C6}
$$

$$\begin{aligned}&-m_0\int_0^L v_{tt}^n\phi_2^n\mathrm{d}z-EI\int_0^L v_{zz}^n\phi_{2zz}^n\mathrm{d}z-P_0\int_0^L v_z^n\phi_{2z}^n\mathrm{d}z-\\&\frac{EA}{2}\int_0^L v_z^{3n}\phi_{2z}^n\mathrm{d}z-\Omega_{1D}\int_0^L v_t^n\phi_2^n\mathrm{d}z-\frac{EA}{2}\int_0^L v_z^n u_z^{2n}\phi_{2z}^n\mathrm{d}z-\\&EA\int_0^L w_z^n v_z^n\phi_{2z}^n\mathrm{d}z+\int_0^L f_v^n v^n\mathrm{d}z-[k_1v^n(L,t)+k_2v_t^n(L,t)]\\&\phi_2^n(L,t)=0\end{aligned}\tag{C7}$$

$$\begin{aligned}&-m_0\int_0^L w_{tt}^n\phi_3^n\mathrm{d}z+EA\int_0^L w_z^n\phi_{3z}^n\mathrm{d}z-\frac{EA}{2}\int_0^L u_z^{2n}\phi_{3z}^n\mathrm{d}z-\\&\frac{EA}{2}\int_0^L v_z^{2n}\phi_{3z}^n\mathrm{d}z-\Omega_{3D}\int_0^L w_t^n\phi_3^n\mathrm{d}z+\int_0^L f_w^n\phi_3^n\mathrm{d}z-\\&[k_5w^n(L,t)+k_6w_t^n(L,t)]\phi_3^n(L,t)=0\end{aligned}\tag{C8}$$

**Estimate 1:** Upper bounds of $\int_0^L\left(u_t^2+v_t^2+w_t^2\right)\mathrm{d}z$, and $\int_0^L\left(u_z^2+v_z^2+w_z^2\right)\mathrm{d}z+\int_0^L\left(u_{zz}^2+v_{zz}^2\right)\mathrm{d}z$

Taking $\phi_1=u_t^n,\phi_2=v_t^n$, and $\phi_3=w_t^n$ in (C.3), (C.4), and (C.5), respectively, leads to:

$$\begin{aligned}&-m_0\int_0^L u_{tt}^n u_t^n\mathrm{d}z-EI\int_0^L u_{zz}^n u_{zzt}^n\mathrm{d}z-P_0\int_0^L u_z^n u_{zt}^n\mathrm{d}z-\\&\frac{EA}{2}\int_0^L u_z^{3n}u_{zt}^n\mathrm{d}z-\Omega_{1D}\int_0^L u_t^{2n}\mathrm{d}z-\frac{EA}{2}\int_0^L u_z^n v_z^{2n}u_{zt}^n\mathrm{d}z-\\&EA\int_0^L w_z^n u_z^n u_{zt}^n\mathrm{d}z+\int_0^L f_u^n u_t^n\mathrm{d}z-[k_1u^n(L,t)+k_2u_t^n(L,t)]\\&u_t^n(L,t)=0\end{aligned}\tag{C9}$$

$$\begin{aligned}&-m_0\int_0^L v_{tt}^n v_t^n\mathrm{d}z-EI\int_0^L v_{zz}^n u_{zzt}^n\mathrm{d}z-P_0\int_0^L v_z^n u_{zt}^n\mathrm{d}z-\\&\frac{EA}{2}\int_0^L v_z^{3n}u_{zt}^n\mathrm{d}z-\Omega_{1D}\int_0^L v_t^{2n}\mathrm{d}z-\frac{EA}{2}\int_0^L v_z^n u_z^{2n}v_{zt}^n\mathrm{d}z-\\&EA\int_0^L w_z^n v_z^n v_{zt}^n\mathrm{d}z+\int_0^L f_v^n v_t^n\mathrm{d}z-[k_1v^n(L,t)+k_2v_t^n(L,t)]\\&v_t^n(L,t)=0\end{aligned}\tag{C10}$$

$$\begin{aligned}&-m_0\int_0^L w_{tt}^n w_t^n\mathrm{d}z+EA\int_0^L w_z^n w_{zt}^n\mathrm{d}z-\frac{EA}{2}\int_0^L u_z^{2n}w_{zt}^n\mathrm{d}z-\\&\frac{EA}{2}\int_0^L v_z^{2n}w_{zt}^n\mathrm{d}z-\Omega_{3D}\int_0^L w_t^{2n}\mathrm{d}z+\int_0^L f_w^n w_t^n\mathrm{d}z-\\&[k_5w^n(L,t)+k_6w_t^n(L,t)]w_t^n(L,t)=0\end{aligned}\tag{C11}$$

$$\begin{aligned}V_n=&\frac{m_0}{2}\int_0^L\left(u_t^{2n}+v_t^{2n}+w_t^{2n}\right)\mathrm{d}z+\frac{P_0}{2}\int_0^L\left(u_z^{2n}+v_z^{2n}\right)\mathrm{d}z+\\&\frac{EA}{2}\int_0^L w_z^{2n}\mathrm{d}z+\frac{EA}{2}\int_0^L\left(w_z^n+\frac{u_z^{2n}}{2}+\frac{v_z^{2n}}{2}\right)^2\mathrm{d}z+\\&\frac{EI}{2}\int_0^L\left(u_{zz}^{2n}+v_{zz}^{2n}\right)\mathrm{d}z+\rho_1\int_0^L u^n u_t^n\mathrm{d}z+\rho_2\int_0^L v^n v_t^n\mathrm{d}z+\\&\rho_3\int_0^L w^n w_t^n\mathrm{d}z+\left(k_1+\frac{k_2\rho_1}{m_0}\right)u^{2n}(L,t)+\\&\left(k_3+\frac{k_4\rho_2}{m_0}\right)v^{2n}(L,t)+\left(k_5+\frac{k_6\rho_3}{m_0}\right)w^{2n}(L,t)\end{aligned}\tag{C12}$$

where $\rho_1$, $\rho_2$ and $k_i$ for $i$ = 1,…,4, are positive constants. It is apparent, as in Section 3, $V_n$ is a proper function. Differentiating (C12) along the solution of (C3), (C4), and (C5), and using the same technique as in Section 2 lead to:

$$\begin{aligned}\dot V_n\le&-k_1\frac{\rho_1}{m_0}u^{2n}(L,t)-k_2u_t^{2n}(L,t)-k_3\frac{\rho_2}{m_0}v^{2n}(L,t)-\\&k_4v_t^{2n}(L,t)-\frac{k_5\rho_3}{m_0}w^{2n}(L,t)-k_6w_t^{2n}(L,t)-\\&cV_n+\Delta_{cn}\end{aligned}\tag{C13}$$

where

$$\begin{aligned}\Delta_{cn}=&\frac{\rho_1}{m_0}\int_0^L u^n f_u^n\mathrm{d}z+\frac{\rho_2}{m_0}\int_0^L v^n f_v^n\mathrm{d}z+\frac{\rho_3}{m_0}\int_0^L w^n f_w^n\mathrm{d}z+\\&\int_0^L v_z^n f_v\mathrm{d}z+\int_0^L u_t^n f_u^n\mathrm{d}z+\int_0^L w_t^n f_w^n\mathrm{d}z\end{aligned}\tag{C14}$$

An upper bound of $\Delta_{cn}$ can be written as

$$\begin{aligned}\Delta_{cn}\le&\frac{1}{\Upsilon_7}\int_0^L u_t^{2n}\mathrm{d}z+\Upsilon_7\int_0^L f_u^{2n}\mathrm{d}z+\frac{4L^2\rho_1}{m_0\Upsilon_8}\int_0^L u_z^{2n}\mathrm{d}z+\\&\frac{\Upsilon_8\rho_1}{m_0}\int_0^L f_u^{2n}\mathrm{d}z+\frac{1}{\Upsilon_9}\int_0^L v_t^{2n}\mathrm{d}z+\Upsilon_9\int_0^L f_v^{2n}\mathrm{d}z+\\&\frac{4L^2\rho_2}{m_0\Upsilon_{10}}\int_0^L v_z^{2n}\mathrm{d}z+\frac{\Upsilon_{10}\rho_2}{m_0}\int_0^L f_v^{2n}\mathrm{d}z+\frac{1}{\Upsilon_{11}}\int_0^L w_t^{2n}\mathrm{d}z+\\&\Upsilon_{11}\int_0^L f_w^{2n}\mathrm{d}z+\frac{4L^2\rho_3}{m_0\Upsilon_{12}}\int_0^L w_z^{2n}\mathrm{d}z+\frac{\Upsilon_{12}\rho_3}{m_0}\int_0^L f_w^{2n}\mathrm{d}z\end{aligned}\tag{C15}$$

There exists a strictly positive constant $\xi$ such that the following inequality holds

$$\begin{aligned}\Delta_{cn}\le&\xi\left(\int_0^L u_z^{2n}\mathrm{d}z+\int_0^L u_t^{2n}\mathrm{d}z+\int_0^L v_z^{2n}\mathrm{d}z+\int_0^L v_t^{2n}\mathrm{d}z+\right.\\&\left.\int_0^L w_z^{2n}\mathrm{d}z+\int_0^L w_t^{2n}\mathrm{d}z\right)+\frac{1}{\xi}\left(\Upsilon_7+\frac{\Upsilon_8\rho_1}{m_0}\right)\int_0^L f_u^{2n}\mathrm{d}z+\\&\frac{1}{\xi}\left(\Upsilon_9+\frac{\Upsilon_{10}\rho_2}{m_0}\right)\int_0^L f_v^{2n}\mathrm{d}z+\frac{1}{\xi}\left(\Upsilon_{11}+\frac{\Upsilon_{12}\rho_3}{m_0}\right)\\&\int_0^L f_w^{2n}\mathrm{d}z\end{aligned}\tag{C16}$$

From the lower bound of $V$, it is shown that

$$\begin{aligned}&\xi\left(\int_0^L u_z^{2n}\mathrm{d}z+\int_0^L u_t^{2n}\mathrm{d}z+\int_0^L v_z^{2n}\mathrm{d}z+\right.\\&\left.\int_0^L v_t^{2n}\mathrm{d}z+\int_0^L w_z^{2n}\mathrm{d}z+\int_0^L w_t^{2n}\mathrm{d}z\right)\le\xi\frac{V}{\zeta}\end{aligned}\tag{C17}$$

where

$$\begin{aligned}\zeta=\min\Bigg\{&c_1,c_2,c_3,c_4,c_5,c_6,\frac{EA}{2},\frac{EI}{2},\frac{1}{2}\left(k_1+\frac{k_2\rho_1}{m_0}\right),\\&\frac{1}{2}\left(k_3+\frac{k_4\rho_2}{m_0}\right),\frac{1}{2}\left(k_5+\frac{k_6\rho_3}{m_0}\right)\Bigg\}\end{aligned}\tag{C18}$$

Substituting (C82) and (C83) into (C79) gives

$$\dot{V}_n \le -k_1\frac{\rho_1}{m_0}u^2(L,t)-k_2u_t^{2n}(L,t)-k_3\frac{\rho_2}{m_0}v^{2n}(L,t)-$$
$$k_4v_t^{2n}(L,t)-\frac{k_5\rho_3}{m_0}w^2(L,t)-k_6w_t^{2n}(L,t)-\left(c-\frac{\xi}{\zeta}\right)V_n+\frac{1}{\xi}Q_n \tag{C19}$$

where

$$Q_n=\left(\Upsilon_7+\frac{\Upsilon_8\rho_1}{m_0}\right)Q_{1n}+\left(\Upsilon_9+\frac{\Upsilon_{10}\rho_2}{m_0}\right)Q_{2n}+\left(\Upsilon_{11}+\frac{\Upsilon_{12}\rho_3}{m_0}\right)Q_{3n} \tag{C20}$$

and

$$Q_{1n}=\max_{t\ge 0}\int_0^L f_u^{2n}\mathrm{d}z,\quad Q_{2n}=\max_{t\ge 0}\int_0^L f_v^{2n}\mathrm{d}z$$
$$Q_{3n}=\max_{t\ge 0}\int_0^L f_w^{2n}\mathrm{d}z \tag{C21}$$

If $\xi$ is selected in such a way that $\bar{c}=c-\frac{\xi}{\zeta}$ is strictly positive, then:

$$\dot{V}_n \le -\bar{c}V_n+\frac{1}{\xi}Q_n \tag{C22}$$

Multiplying both sides of the above equation with $\mathrm{e}^{\bar{c}t}$ and integrating the resulting equation give

$$V_n(t)\le\left(V_n(t_0)+\frac{1}{\xi}Q_n\right)\mathrm{e}^{-\bar{c}(t-t_0)}+\frac{1}{\xi}Q_n \tag{C23}$$

From (C23), it can be deduced that there exists a nonnegative constant $M_1$ such that:

$$\int_0^L\left(u_t^{2n}+v_t^{2n}+w_t^{2n}\right)\mathrm{d}z+\int_0^L\left(u_z^{2n}+v_z^{2n}+w_z^{2n}\right)\mathrm{d}z+$$
$$\int_0^L\left(u_{zz}^{2n}+v_{zz}^{2n}\right)\mathrm{d}z\le M_1\quad\forall t\in[0,T],n\in N \tag{C24}$$

**Estimate 2:** Upper bounds of $\int_0^L u_{tt}^{2n}(z,t_0)\mathrm{d}z$ , $\int_0^L v_{tt}^{2n}(z,t_0)\mathrm{d}z$ and $\int_0^L w_{tt}^{2n}(z,t_0)\mathrm{d}z$ in the $L^2$-norm

Taking $\phi_1=u_{tt}^n(z,t_0)$ , $\phi_2=v_{tt}^n(z,t_0)$ and $\phi_3=w_{tt}^n(z,t_0)$ in (C.3), (C.4), and (C.5), respectively, results in

$$\begin{aligned}&-m_0\int_0^L u_{tt}^{2n}(z,t_0)\mathrm{d}z-EI\int_0^L u_{zz}^n(z,t_0)u_{zztt}^n(z,t_0)\mathrm{d}z-\\&\frac{EA}{2}\int_0^L u_z^n(z,t_0)v_z^{2n}(z,t_0)u_{ztt}^n(z,t_0)\mathrm{d}z-\\&\frac{EA}{2}\int_0^L u_z^{3n}(z,t_0)u_{ztt}^n(z,t_0)\mathrm{d}z-P_0\int_0^L u_z^n(z,t_0)u_{ztt}^n(z,t_0)\mathrm{d}z-\\&EA\int_0^L w_z^n(z,t_0)u_z^n(z,t_0)u_{ztt}^n(z,t_0)\mathrm{d}z-\\&\Omega_D\int_0^L u_t^n(z,t_0)u_{tt}^n(z,t_0)\mathrm{d}z+\int_0^L f_u^n(z,t_0)u_{tt}^n(z,t_0)\mathrm{d}z-\\&\left[k_1u^n(L,t_0)+k_2u_t^n(L,t_0)\right]u_{tt}^n(L,t_0)=0\end{aligned} \tag{C25}$$

$$\begin{aligned}&-m_0\int_0^L v_{tt}^{2n}(z,t_0)\mathrm{d}z-EI\int_0^L v_{zz}^n(z,t_0)v_{zztt}^n(z,t_0)\mathrm{d}z-\\&\frac{EA}{2}\int_0^L v_z^n(z,t_0)u_z^{2n}(z,t_0)v_{ztt}^n(z,t_0)\mathrm{d}z-\\&\frac{EA}{2}\int_0^L v_z^{3n}(z,t_0)v_{ztt}^n(z,t_0)\mathrm{d}z-\\&P_0\int_0^L v_z^n(z,t_0)v_{ztt}^n(z,t_0)\mathrm{d}z-\\&EA\int_0^L w_z^n(z,t_0)v_z^n(z,t_0)v_{ztt}^n(z,t_0)\mathrm{d}z-\\&\Omega_{2D}\int_0^L v_t^n(z,t_0)v_{tt}^n(z,t_0)\mathrm{d}z+\int_0^L f_v^n(z,t_0)v_{tt}^n(z,t_0)\mathrm{d}z-\\&\left[k_1v^n(L,t_0)+k_2v_t^n(L,t_0)\right]v_{tt}^n(L,t_0)=0\end{aligned} \tag{C26}$$

$$\begin{aligned}&-m_0\int_0^L w_{tt}^{2n}(z,t_0)\mathrm{d}z+EA\int_0^L w_z^n(z,t_0)w_{ztt}^n(z,t_0)\mathrm{d}z-\\&\frac{EA}{2}\int_0^L u_z^{2n}(z,t_0)w_{ztt}^n(z,t_0)\mathrm{d}z-\\&\frac{EA}{2}\int_0^L v_z^{2n}(z,t_0)w_{ztt}^n(z,t_0)\mathrm{d}z-\\&\Omega_{3D}\int_0^L w_z^n(z,t_0)w_{tt}^n(z,t_0)\mathrm{d}z+\\&\int_0^L f_w^n(z,t_0)w_{tt}^n(z,t_0)\mathrm{d}z-\\&\left[k_5w^n(L,t_0)+k_6w_t^n(L,t_0)\right]w_{tt}^n(L,t_0)=0\end{aligned} \tag{C27}$$

Integrating (C25), (C26), and (C27) by parts and applying compatibility conditions $-EIu_{zzz}^n(L,t_0)+P_0u_z^n(L,t_0)+\frac{EA}{2}u_z^{3n}(L,t_0)+EAw_z^n(L,t_0)u_z^n(L,t_0)+\frac{EA}{2}u_z^n(L,t_0)v_z^{2n}(L,t_0)=-k_1u^n(L,t_0)-k_2u_t^n(L,t_0)$ , $-EIv_{zzz}^n(L,t_0)+P_0v_z^n(L,t_0)+\frac{EA}{2}v_z^{3n}(L,t_0)+EAw_z^n(L,t_0)v_z^n(L,t_0)+\frac{EA}{2}v_z^n(L,t_0)u_z^{2n}(L,t_0)=-k_3v^n(L,t_0)-k_4v_t^n(L,t_0)$ , and $EAw_z^n(L,t_0)+\frac{EA}{2}u_z^{2n}(L,t_0)+\frac{EA}{2}v_z^{2n}(L,t_0)=-k_5w^n(L,t_0)-k_6w_t^n(L,t_0)$ , and the boundary conditions result in

$$\begin{aligned}&m_0\int_0^L u_{tt}^{2n}(z,t_0)\mathrm{d}z-EI\int_0^L u_{zzzz}^n(z,t_0)u_{tt}^n(z,t_0)\mathrm{d}z+\\&P_0\int_0^L u_{zz}^n(z,t_0)u_{tt}^n(z,t_0)\mathrm{d}z+\\&\frac{3EA}{2}\int_0^L u_{tt}^n(z,t_0)u_z^{2n}(z,t_0)u_{zz}^u(z,t_0)\mathrm{d}z+\\&\frac{EA}{2}\int_0^L u_{tt}^n(z,t_0)u_{zz}^n(z,t_0)v_z^{2n}(z,t_0)\mathrm{d}z+\\&EA\int_0^L v_z^n(z,t_0)v_{zz}^n(z,t_0)u_z^n(z,t_0)u_{tt}^n(z,t_0)\mathrm{d}z+\\&EA\int_0^L w_{zz}^n(z,t_0)u_z^n(z,t_0)u_{tt}^n(z,t_0)\mathrm{d}z+\\&EA\int_0^L w_{zz}^n(z,t_0)u_{zz}^n(z,t_0)u_{tt}^n(z,t_0)\mathrm{d}z-\\&\Omega_D\int_0^L u_t^n(z,t_0)u_{tt}^n(z,t_0)\mathrm{d}z+\int_0^L f_u^n(z,t_0)u_{tt}^n(z,t_0)\mathrm{d}z=0\end{aligned} \tag{C28}$$

$$\begin{aligned}
&m_0\int_0^L v_{tt}^{2n}(z,t_0)\mathrm{d}z - EI\int_0^L v_{zzzz}^{n}(z,t_0)v_{tt}^{n}(z,t_0)\mathrm{d}z + \\
&P_0\int_0^L v_{zz}^{n}(z,t_0)v_{tt}^{n}(z,t_0)\mathrm{d}z + \\
&\frac{3EA}{2}\int_0^L v_{tt}^{n}(z,t_0)v_{z}^{2n}(z,t_0)v_{zz}^{n}(z,t_0)\mathrm{d}z + \\
&\frac{EA}{2}\int_0^L v_{tt}^{n}(z,t_0)v_{zz}^{n}(z,t_0)u_{z}^{2n}(z,t_0)\mathrm{d}z + \\
&EA\int_0^L u_{z}^{n}(z,t_0)u_{zz}^{n}(z,t_0)v_{z}^{n}(z,t_0)v_{tt}^{n}(z,t_0)\mathrm{d}z + \\
&EA\int_0^L w_{zz}^{n}(z,t_0)v_{z}^{n}(z,t_0)v_{tt}^{n}(z,t_0)\mathrm{d}z + \\
&EA\int_0^L w_{zz}^{n}(z,t_0)v_{zz}^{n}(z,t_0)v_{tt}^{n}(z,t_0)\mathrm{d}z - \\
&\Omega_{2D}\int_0^L v_{t}^{n}(z,t_0)v_{tt}^{n}(z,t_0)\mathrm{d}z + \int_0^L f_{v}^{n}(z,t_0)v_{tt}^{n}(z,t_0)\mathrm{d}z = 0
\end{aligned} \tag{C29}$$

$$\begin{aligned}
&-m_0\int_0^L w_{tt}^{2n}(z,t_0)\mathrm{d}z - EA\int_0^L w_{zz}^{n}(z,t_0)w_{tt}^{n}(z,t_0)\mathrm{d}z + \\
&EA\int_0^L u_{z}^{n}(z,t_0)u_{zz}^{n}(z,t_0)w_{tt}^{n}(z,t_0)\mathrm{d}z + \\
&\int_0^L EAv_{z}^{n}(z,t_0)v_{zz}^{n}(z,t_0)w_{tt}^{n}(z,t_0)\mathrm{d}z - \\
&\Omega_{3D}\int_0^L w_{t}^{n}(z,t_0)w_{tt}^{n}(z,t_0)\mathrm{d}z + \int_0^L f_{w}(z,t_0)w_{tt}^{n}(z,t_0)\mathrm{d}z = 0
\end{aligned} \tag{C30}$$

From (C28), (C29), and (C30), it is shown that

$$\begin{aligned}
&(m_0-9\mu_1)\int_0^L u_{tt}^{2n}(z,t_0)\mathrm{d}z \le \frac{(EI)^2}{\mu_1}\int_0^L u_{zzzz}^{2n}(z,t_0)\mathrm{d}z + \\
&\frac{P_0^2}{\mu_1}\int_0^L u_{zz}^{2n}(z,t_0)\mathrm{d}z - \frac{\Omega_{1D}^2}{\mu_1}\int_0^L u_{t}^{2n}(z,t_0)\mathrm{d}z + \\
&\left(\frac{3EA}{2}\right)^2\frac{1}{\mu_1}\int_0^L (z,t_0)u_{z}^{4n}(z,t_0)u_{zz}^{2n}(z,t_0)\mathrm{d}z + \\
&\left(\frac{EA}{2}\right)^2\frac{1}{\mu_1}\int_0^L u_{zz}^{2n}(z,t_0)v_{z}^{4n}(z,t_0)\mathrm{d}z + \\
&\frac{(EA)^2}{\mu_1}\int_0^L v_{z}^{2n}(z,t_0)v_{zz}^{2n}(z,t_0)u_{z}^{2n}(z,t_0)\mathrm{d}z + \\
&\frac{(EA)^2}{\mu_1}\int_0^L w_{zz}^{2n}(z,t_0)u_{z}^{n}(z,t_0)\mathrm{d}z + \\
&\frac{(EA)^2}{\mu_1}\int_0^L w_{zz}^{2n}(z,t_0)u_{zz}^{n}(z,t_0)\mathrm{d}z + \frac{1}{\mu_1}\int_0^L f_{u}^{2n}(z,t_0)\mathrm{d}z = 0
\end{aligned} \tag{C31}$$

$$\begin{aligned}
&(m_0-9\mu_2)\int_0^L v_{tt}^{2n}(z,t_0)\mathrm{d}z \le \frac{(EI)^2}{\mu_2}\int_0^L v_{zzzz}^{2n}(z,t_0)\mathrm{d}z + \\
&\frac{P_0^2}{\mu_2}\int_0^L v_{zz}^{2n}(z,t_0)\mathrm{d}z - \frac{\Omega_{1D}^2}{\mu_1}\int_0^L v_{t}^{2n}(z,t_0)\mathrm{d}z + \\
&\left(\frac{3EA}{2}\right)^2\frac{1}{\mu_2}\int_0^L v_{z}^{4n}(z,t_0)v_{zz}^{2n}(z,t_0)\mathrm{d}z + \\
&\left(\frac{EA}{2}\right)^2\frac{1}{\mu_2}\int_0^L v_{zz}^{2n}(z,t_0)u_{z}^{4n}(z,t_0)\mathrm{d}z + \\
&\frac{(EA)^2}{\mu_2}\int_0^L u_{z}^{2n}(z,t_0)u_{zz}^{2n}(z,t_0)v_{z}^{2n}(z,t_0)\mathrm{d}z + \\
&\frac{(EA)^2}{\mu_2}\int_0^L w_{zz}^{2n}(z,t_0)v_{z}^{2n}(z,t_0)\mathrm{d}z + \\
&\frac{(EA)^2}{\mu_2}\int_0^L w_{zz}^{2n}(z,t_0)v_{zz}^{2n}(z,t_0)\mathrm{d}z + \frac{1}{\mu_2}\int_0^L f_{v}^{2n}(z,t_0)\mathrm{d}z = 0
\end{aligned} \tag{C32}$$

$$\begin{aligned}
&(m_0-5\mu_3)\int_0^L w_{tt}^{2n}(z,t_0)\mathrm{d}z \le \frac{(EA)^2}{\mu_3}\int_0^L w_{zz}^{2n}(z,t_0)\mathrm{d}z + \\
&\frac{(EA)^2}{\mu_3}\int_0^L u_{z}^{2n}(z,t_0)u_{zz}^{2n}(z,t_0)\mathrm{d}z + \\
&\frac{(EA)^2}{\mu_3}\int_0^L v_{z}^{2n}(z,t_0)v_{zz}^{2n}(z,t_0)\mathrm{d}z - \\
&\frac{\Omega_{3D}^2}{\mu_3}\int_0^L w_{t}^{2n}(z,t_0)\mathrm{d}z + \frac{1}{\mu_3}\int_0^L f_{w}^{2n}(z,t_0)\mathrm{d}z = 0
\end{aligned} \tag{C33}$$

where $\mu_1$, $\mu_2$, and $\mu_3$ are strictly positive constants. We have provided the boundedness of $\int_0^L\left(u_t^{2n}+v_t^{2n}+w_t^{2n}\right)\mathrm{d}z$, $\int_0^L\left(u_z^{2n}+v_z^{2n}+w_z^{2n}\right)\mathrm{d}z$ and $\int_0^L\left(u_{zz}^{2}+v_{zz}^{2}\right)\mathrm{d}z$ and since the initial values $u(z,t_0)$, $v(z,t_0)$, $w(z,t_0)$, $u_t(z,t_0)$, $v_t(z,t_0)$, and $w(z,t_0)$ are sufficiently smooth, from (C31), (C32), and (C33), it can be concluded that:

$$\int_0^L u_{tt}^{2n}(z,t_0)\mathrm{d}z \le M_2 \tag{C34}$$

$$\int_0^L v_{tt}^{2n}(z,t_0)\mathrm{d}z \le M_3 \tag{C35}$$

$$\int_0^L w_{tt}^{2n}(z,t_0)\mathrm{d}z \le M_4 \tag{C36}$$

for all $t \in [0,\ T]$, $n \in N$, where $M_2$, $M_3$, and $M_4$ are nonnegative constants, provided that $\mu_1$ and $\mu_2$ are chosen strictly less than $\frac{m_0}{9}$, and $\mu_3$ is chosen strictly less than $\frac{m_0}{5}$.

**Estimate 3:** Upper bounds of $\int_0^L u_{tt}^{2n}(z,t)\mathrm{d}z$, $\int_0^L v_{tt}^{2n}(z,t)\mathrm{d}z$, and $\int_0^L w_{tt}^{2n}(z,t)\mathrm{d}z$ in the $L^2$-norm

Let's fix $t$, $\xi>0$ such that $\xi<T-t$. Taking the difference of (C3), (C4), and (C5) with $t=t+\xi$ and $t=t_0$, and simultaneously replacing $\phi_1$, $\phi_2$, and $\phi_3$ with $u_t^n(t+\xi)-u_t^n(t)$, $v_t^n(t+\xi)-v_t^n(t)$, and $w_t^n(t+\xi)-w_t^n(t)$, respectively, lead to

$$\begin{aligned}
&\frac{m_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left[u_t^n(z,t+\xi)-u_t^n(z,t)\right]^2\mathrm{d}z + \\
&\frac{EI}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left[u_{zz}^n(z,t+\xi)-u_{zz}^n(z,t)\right]^2\mathrm{d}z + \Omega_1 = 0
\end{aligned} \tag{C37}$$

where

$$\begin{aligned}
&\Omega_1 = P_0\int_0^L\left[u_z^n(z,t+\xi)-u_z^n(z,t)\right]\left[u_{xt}^n(z,t+\xi)-u_{xt}^n(z,t)\right]\mathrm{d}z + \\
&\frac{EA}{2}\int_0^L\left[u_z^n(z,t+\xi)-u_z^n(z,t)\right]^3\left[u_{xt}^n(z,t+\xi)-u_{xt}^n(z,t)\right]\mathrm{d}z + \\
&\Omega_{1D}\int_0^L\left[u_t^n(z,t+\xi)-u_t^n(z,t)\right]^2\mathrm{d}z + \\
&\frac{EA}{2}\int_0^L\left[u_z^n(z,t+\xi)-u_z^n(z,t)\right]\left[v_z^n(z,t+\xi)-v_z^n(z,t)\right]^2 \\
&\left[u_{xt}^n(z,t+\xi)-u_{xt}^n(z,t)\right]\mathrm{d}z - \int_0^L\left[f_u^n(z,t+\xi)-f_u^n(z,t)\right] \\
&\left[u_t^n(z,t+\xi)-u_t^n(z,t)\right]\mathrm{d}z + \\
&EA\int_0^L\left[w_z^n(z,t+\xi)-w_z^n(z,t)\right]\left[u_z^n(z,t+\xi)-u_z^n(z,t)\right] \\
&\left[u_{zt}^n(z,t+\xi)-u_{zt}^n(z,t)\right]\mathrm{d}z + \left[\left(k_1u^n(L,t+\xi)-k_1u^n(L,t)\right)+\right. \\
&\left.\left(k_2u_t^n(L,t+\xi)-k_2u_t^n(L,t)\right)\right]\left[u_t^n(L,t+\xi)-u_t^n(L,t)\right]
\end{aligned} \tag{C38}$$

$$\begin{aligned}&\frac{m_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left[v_t^n\left(z,t+\xi\right)-v_t^n\left(z,t\right)\right]^2\mathrm{d}z+\\&\frac{EI}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left[v_{zz}^n\left(z,t+\xi\right)-v_{zz}^n\left(z,t\right)\right]^2\mathrm{d}z+\varOmega_2=0\end{aligned}\tag{C39}$$

where

$$\begin{aligned}&\varOmega_2=P_0\int_0^L\left[v_z^n\left(z,t+\xi\right)-v_z^n\left(z,t\right)\right]\\&\left[v_{xt}^n\left(z,t+\xi\right)-v_{xt}^n\left(z,t\right)\right]\mathrm{d}z+\\&\frac{EA}{2}\int_0^L\left[v_z^n\left(z,t+\xi\right)-v_z^n\left(z,t\right)\right]^3\left[v_{xt}^n\left(z,t+\xi\right)-v_{xt}^n\left(z,t\right)\right]\mathrm{d}z+\\&\varOmega_{1D}\int_0^L\left[v_t^n\left(z,t+\xi\right)-v_t^n\left(z,t\right)\right]^2\mathrm{d}z+\\&\frac{EA}{2}\int_0^L\left[v_z^n\left(z,t+\xi\right)-v_z^n\left(z,t\right)\right]\\&\left[u_z^n\left(z,t+\xi\right)-u_z^n\left(z,t\right)\right]^2\left[v_{xt}^n\left(z,t+\xi\right)-v_{xt}^n\left(z,t\right)\right]\mathrm{d}z-\\&\int_0^L\left[f_v\left(z,t+\xi\right)-f_v\left(z,t\right)\right]\left[v_t^n\left(z,t+\xi\right)-v_t^n\left(z,t\right)\right]\mathrm{d}z+\\&EA\int_0^L\left[w_z^n\left(z,t+\xi\right)-w_z^n\left(z,t\right)\right]\left[v_z^n\left(z,t+\xi\right)-v_z^n\left(z,t\right)\right]\\&\left[v_{zt}^n\left(z,t+\xi\right)-v_{zt}^n\left(z,t\right)\right]\mathrm{d}z+\left[\left(k_3v^n\left(L,t+\xi\right)-k_3v^n\left(L,t\right)\right)+\right.\\&\left.\left(k_4v_t^n\left(L,t+\xi\right)-k_4v_t^n\left(L,t\right)\right)\right]\left[v_t^n\left(L,t+\xi\right)-v_t^n\left(L,t\right)\right]\end{aligned}\tag{C40}$$

and

$$\begin{aligned}&\frac{m_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left[w_t^n\left(z,t+\xi\right)-w_t^n\left(z,t\right)\right]^2\mathrm{d}z+\\&EA\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left[w_z^n\left(z,t+\xi\right)-w_z^n\left(z,t\right)\right]^2\mathrm{d}z+\varOmega_3=0\end{aligned}\tag{C41}$$

where

$$\begin{aligned}&\varOmega_3=\frac{EA}{2}\int_0^L\left[u_z^{2n}\left(z,t_0+\xi\right)-u_z^{2n}\left(z,t_0\right)\right]\\&\left[w_{zt}^n\left(z,t_0+\xi\right)-w_{zt}^n\left(z,t_0\right)\right]\mathrm{d}z+\\&\frac{EA}{2}\int_0^L\left[v_z^{2n}\left(z,t_0+\xi\right)-v_z^{2n}\left(z,t_0\right)\right]\\&\left[w_{zt}^n\left(z,t_0+\xi\right)-w_{zt}^n\left(z,t_0\right)\right]\mathrm{d}z+\\&\varOmega_{1D}\int_0^L\left[w_t^n\left(z,t+\xi\right)-w_t^n\left(z,t\right)\right]^2\mathrm{d}z-\\&\int_0^L\left[f_w\left(z,t+\xi\right)-f_w\left(z,t\right)\right]\left[w_t^n\left(z,t+\xi\right)-w_t^n\left(z,t\right)\right]\mathrm{d}z+\\&\left[\left(k_5w^n\left(L,t+\xi\right)-k_5w^n\left(L,t\right)\right)+\right.\\&\left.\left(k_6w_t^n\left(L,t+\xi\right)-k_6w_t^n\left(L,t\right)\right)\right]\left[w_t^n\left(L,t+\xi\right)-w_t^n\left(L,t\right)\right]\end{aligned}\tag{C42}$$

Integrating $\varOmega_1$, $\varOmega_2$, and $\varOmega_3$ and noting that the initial values $u(z,t_0)$, $u_t(z,t_0)$, $v(z,t_0)$, $v_t(z,t_0)$, $w(z,t_0)$, and $w_t(z,t_0)$ are sufficiently smooth, then $u(0,t)=0$, $u_{zz}(0,t)=0$, $u_{zz}(L,t)=0$, $u(0,t)=0$, $u_{zz}(0,t)=0$, $u_{zz}(L,t)=0$, and $w_z(0,t)=0$ for all $u(z,t)$, $v(z,t)\in W_S$. In addition, $u_t(z,t)$, $v_t(z,t)$, and the spatial derivatives of $u(z,t)$, $v(z,t)$ up to the fourth order are bounded. Using inequalities (A1) and (A4), it can be shown that:

$$\begin{aligned}\left|\varOmega_1\right|\le M_{31}&\int_0^L\left[u_t^n\left(z,t+\xi\right)-u_t^n\left(z,t\right)\right]^2\mathrm{d}z+\\M_{32}&\int_0^L\left[u_{zz}^n\left(z,t+\xi\right)-u_{zz}^n\left(z,t\right)\right]^2\mathrm{d}z\end{aligned}\tag{C43}$$

$$\begin{aligned}\left|\varOmega_2\right|\le M_{33}&\int_0^L\left[v_t^n\left(z,t+\xi\right)-v_t^n\left(z,t\right)\right]^2\mathrm{d}z+\\M_{34}&\int_0^L\left[v_{zz}^n\left(z,t+\xi\right)-v_{zz}^n\left(z,t\right)\right]^2\mathrm{d}z\end{aligned}\tag{C44}$$

$$\begin{aligned}\left|\varOmega_3\right|\le M_{35}&\int_0^L\left[w_t^n\left(z,t+\xi\right)-w_t^n\left(z,t\right)\right]^2\mathrm{d}z+\\M_{36}&\int_0^L\left[u_{zz}^n\left(z,t+\xi\right)-u_{zz}^n\left(z,t\right)\right]^2\mathrm{d}z+\\M_{37}&\int_0^L\left[v_{zz}^n\left(z,t+\xi\right)-v_{zz}^n\left(z,t\right)\right]^2\mathrm{d}z\end{aligned}\tag{C45}$$

where $M_{3i}$, for $i$ = 1,…,7, are nonnegative constants. Applying (C43), (C44), and (C45) to (C37), (C39), and (C41), respectively, yields:

$$\frac{\mathrm{d}\Phi_1^n\left(t,\xi\right)}{\mathrm{d}t}\le M_{38}\varPhi_1^n\left(t,\xi\right)\Rightarrow\varPhi_1\left(t,\xi\right)\le\varPhi_1\left(t_0,\xi\right)\mathrm{e}^{M_{38}\left(t-t_0\right)}\tag{C46}$$

$$\frac{\mathrm{d}\varPhi_2^n\left(t,\xi\right)}{\mathrm{d}t}\le M_{39}\varPhi_2^n\left(t,\xi\right)\Rightarrow\varPhi_2\left(t,\xi\right)\le\varPhi_2\left(t_0,\xi\right)\mathrm{e}^{M_{39}\left(t-t_0\right)}\tag{C47}$$

$$\frac{\mathrm{d}\varPhi_3^n\left(t,\xi\right)}{\mathrm{d}t}\le M_{40}\varPhi_3^n\left(t,\xi\right)\Rightarrow\varPhi_3\left(t,\xi\right)\le\varPhi_3\left(t_0,\xi\right)\mathrm{e}^{M_{40}\left(t-t_0\right)}\tag{C48}$$

where $M_{38}$, $M_{39}$, and $M_{40}$ are nonnegative constants, and:

$$\begin{aligned}\varPhi_1^n\left(t,\xi\right)=m_0&\int_0^L\left[u_t^n\left(z,t+\xi\right)-u_t^n\left(z,t\right)\right]^2\mathrm{d}z+\\EI&\int_0^L\left[u_{zz}^n\left(z,t+\xi\right)-u_{zz}^n\left(z,t\right)\right]^2\mathrm{d}z\end{aligned}\tag{C49}$$

$$\begin{aligned}\varPhi_2^n\left(t,\xi\right)=m_0&\int_0^L\left[v_t^n\left(z,t+\xi\right)-v_t^n\left(z,t\right)\right]^2\mathrm{d}z+\\EI&\int_0^L\left[v_{zz}^n\left(z,t+\xi\right)-v_{zz}^n\left(z,t\right)\right]^2\mathrm{d}z\end{aligned}\tag{C50}$$

$$\begin{aligned}\varPhi_3^n\left(t,\xi\right)=m_0&\int_0^L\left[w_t^n\left(z,t+\xi\right)-w_t^n\left(z,t\right)\right]^2\mathrm{d}z+\\EI&\int_0^L\left[w_z^n\left(z,t+\xi\right)-v_z^n\left(z,t\right)\right]^2\mathrm{d}z\end{aligned}\tag{C51}$$

Dividing both sides of (C46), (C47), and (C48) by $\xi^2$ and taking the limit $\xi\rightarrow0$ lead to

$$\begin{aligned}&m_0\int_0^L u_{tt}^{n2}\left(z,t\right)\mathrm{d}z+EI\int_0^L u_{zzt}^{n2}\left(z,t\right)\mathrm{d}z\le\left[m_0\int_0^L u_{tt}^{n2}\left(z,t_0\right)\mathrm{d}z+\right.\\&\left.EI\int_0^L u_{zzt}^{n2}\left(z,t_0\right)\mathrm{d}z\right]\mathrm{e}^{M_{38}\left(t-t_0\right)}\end{aligned}\tag{C52}$$

$$\begin{aligned}&m_0\int_0^L v_{tt}^{n2}\left(z,t\right)\mathrm{d}z+EI\int_0^L v_{zzt}^{n2}\left(z,t\right)\mathrm{d}z\le\left[m_0\int_0^L v_{tt}^{n2}\left(z,t_0\right)\mathrm{d}z+\right.\\&\left.EI\int_0^L v_{zzt}^{n2}\left(z,t_0\right)\mathrm{d}z\right]\mathrm{e}^{M_{39}\left(t-t_0\right)}\end{aligned}\tag{C53}$$

$$\begin{aligned}&m_0\int_0^L w_{tt}^{n2}(z,t)\mathrm{d}z+EI\int_0^L w_{zzt}^{n2}(z,t)\mathrm{d}z\le\\&\left[m_0\int_0^L w_{tt}^{n2}(z,t_0)\mathrm{d}z+EI\int_0^L w_{zt}^{n2}(z,t_0)\mathrm{d}z\right]\mathrm{e}^{M_{40(t-t_0)}}\end{aligned}\tag{C54}$$

From Estimates 1 and 2, it can be deduced from the above inequalities that there exist nonnegative constants $M_5$, $M_6$, and $M_7$ depending on $T$ such that

$$m_0\int_0^L u_{tt}^{n2}(z,t)\mathrm{d}z+EI\int_0^L u_{zzt}^{n2}(z,t)\mathrm{d}z\le M_5\tag{C55}$$

$$m_0\int_0^L v_{tt}^{n2}(z,t)\mathrm{d}z+EI\int_0^L v_{zzt}^{n2}(z,t)\mathrm{d}z\le M_6\tag{C56}$$

$$m_0\int_0^L w_{tt}^{n2}(z,t)\mathrm{d}z+EI\int_0^L w_{zzt}^{n2}(z,t)\mathrm{d}z\le M_7\tag{C57}$$

By using Estimates 1, 2, and 3 and applying the Lions-Aubin theorem, the nonlinear systems (C6), (C7), and (C8) can be passed to the limit; hence the existence of global solutions is concluded.

### C.2 Proof of uniqueness

Let's define $u$, $v$, $w$ and $\bar{u}$, $\bar{v}$, $\bar{w}$ as two different sets of solutions to the closed-loop system (24). The differences between the two sets are $\theta_1=u-\bar{u}$, $\theta_2=v-\bar{v}$, and $\theta_3=w-\bar{w}$, and it can be seen that $\theta_1(z,t_0)=\theta_2(z,t_0)=\theta_3(z,t_0)=0$ and $\theta_{1t}(z,t_0)=\theta_{2t}(z,t_0)=\theta_{3t}(z,t_0)=0$, and:

$$\begin{aligned}&-m_0\int_0^L\theta_{1tt}\phi_1\mathrm{d}z-EI\int_0^L\theta_{1zz}\phi_{1zz}\mathrm{d}z-P_0\int_0^L\theta_{1z}\phi_{1z}\mathrm{d}z-\\&\frac{EA}{2}\int_0^L\theta_{1z}^3\phi_{1z}\mathrm{d}z-\Omega_D\int_0^L\theta_{1t}\phi_1\mathrm{d}z-\frac{EA}{2}\int_0^L\theta_{1z}\theta_{2z}^2\phi_{1z}\mathrm{d}z-\\&EA\int_0^L\theta_{3z}\theta_{1z}\phi_{1z}\mathrm{d}z+\int_0^L f_u\phi_1\mathrm{d}z-\left[k_1\theta_1(L,t)+k_2\theta_{1t}(L,t)\right]\\&\phi_1(L,t)=0\end{aligned}\tag{C58}$$

$$\begin{aligned}&-m_0\int_0^L\theta_{2tt}\phi_2\mathrm{d}z-EI\int_0^L\theta_{2zz}\phi_{2zz}\mathrm{d}z-P_0\int_0^L\theta_{2z}\phi_{2z}\mathrm{d}z-\\&\frac{EA}{2}\int_0^L\theta_{2z}^3\phi_{2z}\mathrm{d}z-\Omega_{1D}\int_0^L\theta_{2t}\phi_2\mathrm{d}z-\frac{EA}{2}\int_0^L\theta_{2z}\theta_{1z}^2\phi_{2z}\mathrm{d}z-\\&EA\int_0^L\theta_{3z}\theta_{2z}\phi_{2z}\mathrm{d}z+\int_0^L f_v\phi_2\mathrm{d}z-\left[k_3\theta_2(L,t)+k_4\theta_{2t}(L,t)\right]\\&\phi_2(L,t)=0\end{aligned}\tag{C59}$$

$$\begin{aligned}&-m_0\int_0^L\theta_{3tt}\phi_3\mathrm{d}z+EA\int_0^L\theta_{3z}\phi_{3z}\mathrm{d}z-\frac{EA}{2}\int_0^L\theta_{1z}^3\phi_{3z}\mathrm{d}z-\\&\frac{EA}{2}\int_0^L\theta_{2z}^2\phi_{3z}\mathrm{d}z-\Omega_{3D}\int_0^L\theta_{3t}\phi_3\mathrm{d}z-\int_0^L f_w\phi_3\mathrm{d}z-\\&\left[k_5\theta_3(L,t)+k_6\theta_{3t}(L,t)\right]\phi_3(L,t)=0\end{aligned}\tag{C60}$$

Taking $\phi_1=\theta_{1t}(z,t)$, $\phi_2=\theta_{2t}(z,t)$, $\phi_3=\theta_{3t}(z,t)$ in (C58), (C59), and (C60), respectively, and using the same technique in Estimate 3, it can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L\theta_{1t}^2\mathrm{d}z+\int_0^L\theta_{1zz}^2\mathrm{d}z\right]\le M_8\left[\int_0^L\theta_{1t}^2\mathrm{d}z+\int_0^L\theta_{1zz}^2\mathrm{d}z\right]\tag{C61}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L\theta_{2t}^2\mathrm{d}z+\int_0^L\theta_{2zz}^2\mathrm{d}z\right]\le M_9\left[\int_0^L\theta_{2t}^2\mathrm{d}z+\int_0^L\theta_{2zz}^2\mathrm{d}z\right]\tag{C62}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L\theta_{2t}^2\mathrm{d}z+\int_0^L\theta_{2z}^2\mathrm{d}z\right]\le M_{10}\left[\int_0^L\theta_{2t}^2\mathrm{d}z+\int_0^L\theta_{2z}^2\mathrm{d}z\right]\tag{C63}$$

where $M_8$, $M_9$, and $M_{10}$ are positive constants. Using Gronwall's Lemma and the initial conditions indicates that $\theta_1=\theta_2=\theta_3=0$, which implies the uniqueness of the solutions of the system.

### C.3 Proof of convergence

Case 1: Disturbance vectors $f_u=f_v=f_w=0$.

When there are no disturbances, (55) becomes

$$\dot{V}\le-cV\tag{C64}$$

which implies

$$V(t)\le V(t_0)\mathrm{e}^{-c(t-t_0)},\forall t\ge t_0\ge 0\tag{C65}$$

The use of (C65) and the bounds of $V(t)$ given in (35) and (36) show that

$$\begin{aligned}&c_1\int_0^L u_t^2(z,t)\mathrm{d}z+c_2\int_0^L v_t^2(z,t)\mathrm{d}z+c_3\int_0^L w_t^2(z,t)\mathrm{d}z+\\&c_4\int_0^L u_z^2(z,t)\mathrm{d}z+c_5\int_0^L v_z^2(z,t)\mathrm{d}z+c_6\int_0^L w_z^2(z,t)\mathrm{d}z+\\&\frac{EA}{2}\int_0^L\left(w_z(z,t)+\frac{u_z^2(z,t)}{2}+\frac{v_z^2(z,t)}{2}\right)^2\mathrm{d}z+\\&\frac{EI}{2}\int_0^L\left(u_{zz}^2(z,t)+v_{zz}^2(z,t)\right)\mathrm{d}z+\frac{1}{2}\left(k_1+\frac{k_2\rho_1}{m_0}\right)u^2(L,t_0)+\\&\frac{1}{2}\left(k_3+\frac{k_4\rho_2}{m_0}\right)v^2(L,t_0)+\left(k_5+\frac{k_6\rho_3}{m_0}\right)w^2(L,t_0)\le\\&\left[\left(\frac{m_0}{2}+\frac{\rho_1}{\Upsilon_1}\right)\int_0^L u_t^2(z,t_0)\mathrm{d}z+\left(\frac{m_0}{2}+\frac{\rho_2}{\Upsilon_2}\right)\int_0^L v_t^2(z,t_0)\mathrm{d}z+\right.\\&\left(\frac{m_0}{2}+\frac{\rho_3}{\Upsilon_3}\right)\int_0^L w_t^2(z,t_0)\mathrm{d}z+\left(\frac{P_0}{2}+4L^2\Upsilon_1\rho_1\right)\int_0^L u_z^2(z,t_0)\mathrm{d}z+\\&\left(\frac{P_0}{2}+4L^2\Upsilon_2\rho_2\right)\int_0^L v_z^2(z,t_0)\mathrm{d}z+\\&\left(\frac{P_0}{2}+4L^2\Upsilon_3\rho_3\right)\int_0^L w_z^2(z,t_0)\mathrm{d}z+\\&\frac{EA}{2}\int_0^L\left(w_z(z,t)+\frac{u_z^2(z,t_0)}{2}+\frac{v_z^2(z,t_0)}{2}\right)^2\mathrm{d}z+\\&\frac{EI}{2}\int_0^L\left(u_{zz}^2(z,t_0)+v_{zz}^2(z,t_0)\right)\mathrm{d}z+\frac{1}{2}\left(k_1+\frac{k_2\rho_1}{m_0}\right)u^2(L,t_0)+\\&\left.\frac{1}{2}\left(k_3+\frac{k_4\rho_2}{m_0}\right)v^2(L,t_0)+\left(k_5+\frac{k_6\rho_3}{m_0}\right)w^2(L,t_0)\right]\mathrm{e}^{-c(t-t_0)}\end{aligned}\tag{C66}$$

Sufficient smoothness and boundedness of the initial values $u_t(z,t_0)$, $v_t(z,t_0)$, $w_t(z,t_0)$, $u_z(z,t_0)$, $v_z(z,t_0)$, $w_z(z,t_0)$, $u_{zz}(z,t_0)$, and $v_{zz}(z,t_0)$ for all $z\in[0,L]$ imply that the right hand side of (C66) is bounded and exponentially converges to zero. This in turn also guarantees boundedness and exponential convergence of the left hand side of (C66). Subsequently, it can be concluded that $\int_0^L u_t^2 \mathrm{d}z$, $\int_0^L v_t^2 \mathrm{d}z$, $\int_0^L w_t^2 \mathrm{d}z$, $\int_0^L u_z^2 \mathrm{d}z$, $\int_0^L v_z^2 \mathrm{d}z$, $\int_0^L w_z^2 \mathrm{d}z$, $\int_0^L u_{zz}^2 \mathrm{d}z$ and $\int_0^L v_{zz}^2 \mathrm{d}z$ are bounded and exponentially converge to zero. Since $u(0, t) = v(0, t) = w(0, t)=0$, an application of inequality (A2) gives:

$$\int_0^L u^2(z,t)\mathrm{d}z \le 4L^2 \int_0^L u_z^2(z,t)\mathrm{d}z \tag{C67}$$

$$\int_0^L v^2(z,t)\mathrm{d}z \le 4L^2 \int_0^L v_z^2(z,t)\mathrm{d}z \tag{C68}$$

$$\int_0^L w^2(z,t)\mathrm{d}z \le 4L^2 \int_0^L w_z^2(z,t)\mathrm{d}z \tag{C69}$$

It has already proven that $\int_0^L u_z^2(z,t)\mathrm{d}z$, $\int_0^L v_z^2(z,t)\mathrm{d}z$, and $\int_0^L w_z^2(z,t)\mathrm{d}z$ are bounded and exponentially converge to zero, and the above inequalities state that $\int_0^L u^2(z,t)\mathrm{d}z$, $\int_0^L v^2(z,t)\mathrm{d}z$, and $\int_0^L w^2(z,t)\mathrm{d}z$ are also bounded and exponentially converge to zero. Further more, using inequality (A3) yields

$$\max_{z\in[0,L]}\left\{u^2(z,t)\right\} \le 2\sqrt{\int_0^L u^2(z,t)\mathrm{d}z}\sqrt{\int_0^L u_z^2(z,t)\mathrm{d}z} \tag{C70}$$

$$\max_{z\in[0,L]}\left\{v^2(z,t)\right\} \le 2\sqrt{\int_0^L v^2(z,t)\mathrm{d}z}\sqrt{\int_0^L v_z^2(z,t)\mathrm{d}z} \tag{C71}$$

$$\max_{z\in[0,L]}\left\{w^2(z,t)\right\} \le 2\sqrt{\int_0^L w^2(z,t)\mathrm{d}z}\sqrt{\int_0^L w_z^2(z,t)\mathrm{d}z} \tag{C72}$$

Boundedness and exponential convergence of $\int_0^L u^2 \mathrm{d}z$, $\int_0^L v^2 \mathrm{d}z$, and $\int_0^L w^2 \mathrm{d}z$ have been proven, from which it follows that $|u(z,t)|$, $|v(z,t)|$, and $|w(z,t)|$ must be bounded and converge to zero exponentially.

Case 2: Disturbance vectors $f_u, f_v$, and $f_w \neq 0$.

Eq. (55) can be written as

$$\begin{aligned}\dot{V} \le & -k_1\frac{\rho_1}{m_0}u^2(L,t) - k_2 u_t^2(L,t) - k_3\frac{\rho_2}{m_0}v^2(L,t) - \\ & k_4 v_t^2(L,t) - \frac{k_5\rho_3}{m_0}w^2(L,t) - k_6 w_t^2(L,t) - cV + \Delta_c\end{aligned} \tag{C73}$$

where

$$\begin{aligned}\Delta_c = & \frac{\rho_1}{m_0}\int_0^L u f_u \mathrm{d}z + \frac{\rho_2}{m_0}\int_0^L v f_v \mathrm{d}z + \frac{\rho_3}{m_0}\int_0^L w f_w \mathrm{d}z + \\ & \int_0^L v_t f_v \mathrm{d}z + \int_0^L u_t f_u \mathrm{d}z + \int_0^L w_t f_w \mathrm{d}z\end{aligned} \tag{C74}$$

An upper bound of $\Delta_c$ can be written as

$$\begin{aligned}\Delta_c \le & \frac{1}{\Upsilon_7}\int_0^L u_t^2 \mathrm{d}z + \Upsilon_7\int_0^L f_u^2 \mathrm{d}z + \frac{4L^2\rho_1}{m_0\Upsilon_8}\int_0^L u_z^2 \mathrm{d}z + \\ & \frac{\Upsilon_8\rho_1}{m_0}\int_0^L f_u^2 \mathrm{d}z + \frac{1}{\Upsilon_9}\int_0^L v_t^2 \mathrm{d}z + \Upsilon_9\int_0^L f_v^2 \mathrm{d}z + \\ & \frac{4L^2\rho_2}{m_0\Upsilon_{10}}\int_0^L v_z^2 \mathrm{d}z + \frac{\Upsilon_{10}\rho_2}{m_0}\int_0^L f_v^2 \mathrm{d}z + \frac{1}{\Upsilon_{11}}\int_0^L w_t^2 \mathrm{d}z + \\ & \Upsilon_{11}\int_0^L f_w^2 \mathrm{d}z + \frac{4L^2\rho_3}{m_0\Upsilon_{12}}\int_0^L w_z^2 \mathrm{d}z + \frac{\Upsilon_{12}\rho_3}{m_0}\int_0^L f_w^2 \mathrm{d}z\end{aligned} \tag{C75}$$

There exists a strictly positive constant $\xi$ such that the following inequality holds

$$\begin{aligned}\Delta_c \le & \xi\left(\int_0^L u_z^2 \mathrm{d}z + \int_0^L u_t^2 \mathrm{d}z + \int_0^L v_z^2 \mathrm{d}z + \int_0^L v_t^2 \mathrm{d}z + \right. \\ & \left.\int_0^L w_z^2 \mathrm{d}z + \int_0^L w_t^2 \mathrm{d}z\right) + \frac{1}{\xi}\left(\Upsilon_7 + \frac{\Upsilon_8\rho_1}{m_0}\right)\int_0^L f_u^2 \mathrm{d}z + \\ & \frac{1}{\xi}\left(\Upsilon_9 + \frac{\Upsilon_{10}\rho_2}{m_0}\right)\int_0^L f_v^2 \mathrm{d}z + \frac{1}{\xi}\left(\Upsilon_{11} + \frac{\Upsilon_{12}\rho_3}{m_0}\right)\int_0^L f_w^2 \mathrm{d}z\end{aligned} \tag{C76}$$

From the lower bound of $V$, it is shown that

$$\begin{aligned}& \xi\left(\int_0^L u_z^2 \mathrm{d}z + \int_0^L u_t^2 \mathrm{d}z + \int_0^L v_z^2 \mathrm{d}z + \right. \\ & \left.\int_0^L v_t^2 \mathrm{d}z + \int_0^L w_z^2 \mathrm{d}z + \int_0^L w_t^2 \mathrm{d}z\right) \le \xi\frac{V}{\zeta}\end{aligned} \tag{C77}$$

where

$$\begin{aligned}\zeta = \min\Big\{ & c_1, c_2, c_3, c_4, c_5, c_6, \frac{EA}{2}, \frac{EI}{2}, \frac{1}{2}\left(k_1 + \frac{k_2\rho_1}{m_0}\right), \\ & \frac{1}{2}\left(k_3 + \frac{k_4\rho_2}{m_0}\right), \frac{1}{2}\left(k_5 + \frac{k_6\rho_3}{m_0}\right)\Big\}\end{aligned} \tag{C78}$$

Substituting (C82) and (C83) into (C79) gives

$$\begin{aligned}\dot{V} \le & -k_1\frac{\rho_1}{m_0}u^2(L,t) - k_2 u_t^2(L,t) - k_3\frac{\rho_2}{m_0}v^2(L,t) - \\ & k_4 v_t^2(L,t) - \frac{k_5\rho_3}{m_0}w^2(L,t) - k_6 w_t^2(L,t) - \left(c - \frac{\xi}{\zeta}\right)V + \frac{1}{\xi}Q\end{aligned} \tag{C79}$$

where

$$Q = \left(\Upsilon_7 + \frac{\Upsilon_8\rho_1}{m_0}\right)Q_1 + \left(\Upsilon_9 + \frac{\Upsilon_{10}\rho_2}{m_0}\right)Q_2 + \left(\Upsilon_{11} + \frac{\Upsilon_{12}\rho_3}{m_0}\right)Q_3 \tag{C80}$$

and

$$Q_1 = \max_{t\ge 0}\int_0^L f_u^2 \mathrm{d}z,\ Q_2 = \max_{t\ge 0}\int_0^L f_v^2 \mathrm{d}z,\ Q_3 = \max_{t\ge 0}\int_0^L f_w^2 \mathrm{d}z \tag{C81}$$

If $\xi$ is selected in such a way that $\bar{c} = c - \frac{\xi}{\zeta}$ is strictly positive, then:

$$\dot{V} \le -\bar{c}V + \frac{1}{\xi}Q \tag{C82}$$

Multiplying both sides of the above equation with $\mathrm{e}^{\bar{c}t}$ and integrating the resulting equation give

$$V(t) \le \left(V(t_0) + \frac{1}{\xi}Q\right)\mathrm{e}^{\bar{c}(t-t_0)} + \frac{1}{\xi}Q \tag{C83}$$

Inequality (C83) implies that $V(t)$ exponentially converges to the nonnegative constant $\frac{1}{\xi}Q$. It can be deduced that all terms $\int_0^L u_t^2 \mathrm{d}z$, $\int_0^L v_t^2 \mathrm{d}z$, $\int_0^L w_t^2 \mathrm{d}z$, $\int_0^L u_z^2 \mathrm{d}z$, $\int_0^L v_z^2 \mathrm{d}z$, $\int_0^L w_z^2 \mathrm{d}z$, $\int_0^L u_{zz}^2 \mathrm{d}z$, and $\int_0^L v_{zz}^2 \mathrm{d}z$ are bounded and exponentially con- verge to some nonnegative constants less than $\frac{1}{\xi\zeta}Q$. Proof of the boundedness and convergence of $\int_0^L u^2(z,t)\mathrm{d}z$, $\int_0^L v^2(z,t)\mathrm{d}z$, $\int_0^L w^2(z,t)\mathrm{d}z$, $|u(z,t)|$, $|v(z,t)|$, and $|w(z,t)|$ can be carried out in the same way as in the case, in which there are no disturbances.

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