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Sanders' Mid-long Cylindrical Shell Theory and its Application to Ocean Engineering Structures

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Abstract: The cylindrical shell is one of the main structural parts in ocean engineering structures. These cylinders are mostly of medium length, which means that the radius of the cross section is significantly smaller than the length of the cylindrical shell. From the viewpoint of the shell theory, they belong to the mid-long cylindrical shell category. To solve mechanical problems on this kind of structure, especially a cracked cylindrical shell, analysis based on shell theory is necessary. At present the generally used solving system for the mid-long cylindrical shell is too complicated, difficult to solve, and inapplicable to engineering. This paper introduced the Sanders' mid-long cylindrical shell theory which reduces the difficulty of the solution process, and will be suitable for solving problems with complicated boundary conditions. On this basis, the engineering applications of this theory were discussed in conjunction with the problem of a mid-long cylindrical shell having a circumferential crack. The solution process is simple, and the closed form solution can usually be found. In practical engineering applications, it gives satisfactory precision.

Keywords: mid-long cylindrical shell; cylindrical shell theory; circumferential crack; ocean engineering structure
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1 Introduction

As a result of the good comprehensive performance of a cylindrical shell, it usually serves as the supporting structure for various platforms in ocean engineering. Although the safety design standards for this kind of platform are quite strict, historical records show that disastrous events cannot be completely avoided. This is partly due to the fact that cylindrical shells used as supporting structures face direct sea waves surges, which bring a high probability of failure (Li *et al*., 2003; Naser, 2011).

To ensure the safety of ocean engineering structures, various computations of the mechanical characteristics of cylindrical shells are necessary in engineering design. Regarding the method of computational analysis, the most popularly applied method is the finite element method, whose effectiveness has been accepted by the engineering community. Nevertheless, it does not mean that theoretical analysis can be entirely neglected. For finite elements analysis, calculation will be carried out for every specific case and there will be one calculation for one case. In this way, the internal relationships or intrinsic rules among various physical quantities will not be shown clearly. Moreover, as for a structure system with some local defective elements, nonlinear calculations will be inefficient and spend significant resources, which should not be neglected in engineering practice. In this sense, analysis based on shell theory is still necessary.

Cylindrical shells used as supporting structures in ocean engineering belong to the mid-long cylindrical shell category (length l is greater than several times the radius R). As for the theoretical analysis of these kind of problems, the theory of a mid-long cylindrical shell will be crucial. In considering the mid-long cylindrical shell, the basic state of stress is no longer moment free, except that at both ends there is an edge effect or possibly a shallow shell effect (in certain complicated boundary conditions). There will be a kind of semi-membrane state; specifically, the membrane state still remains chiefly along the axial direction of the cylindrical shell, while moment and membrane states exist at the same time along the circumferential direction. By magnitude analysis of major mechanical quantities, these kinds of problems can be solved with the aid of asymptotic expansion according to their specific conditions. In other words, with the solutions of the edge effect equation, the shallow shell equation, and the semi-membrane equation, the approximated analytic solution can be found by using the matched-asymptotic method. However, using such a solution process will inevitably involve complicated mathematical analysis and derivation, making it a difficult and tedious job. Taking the currently adopted governing equation of semi-membrane theory as an example, it is a parabolic partial differential equation with the 8th order circumferential coordinate and the 4th order axial coordinate (Gao and Hwang, 1965; Hwang *et al*., 1988). The difficulty of finding a solution is obvious as the equation itself is very complicated.

In the 1980s, the well-known applied mechanics scholar

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Sanders suggested a set of theories for mid-long cylindrical shells (Sanders, 1959, 1972, 1980a, 1980b, 1983a; Budiansky and Sanders, 1963). The difficulty in solving problems of cylindrical shells is greatly reduced through a series of variable transformations and the derivation of boundary conditions based on the stress functions, making it very suitable to treat problems of cylindrical shells with complicated boundary conditions. This theory reduced to a certain extent the trouble mentioned above and made it applicable to many practical projects.

Since the mid-long cylindrical shell plays a special role in the ocean engineering field, this paper gives a basic introduction to Sanders' mid-long cylindrical shell theory, and discusses its engineering application in conjunction with a problem for a mid-long cylindrical shell having a circumferential crack.

2 Sanders' mid-long cylindrical shell theory

The following parameter ε is introduced

$$\varepsilon^2 = h/R[12(1-\nu^2)]^{-1/2} \tag{1}$$

where h is the shell thickness, R is the radius of the cylinder and ν is Poisson's ratio. One of the reference conditions to define the mid-long cylinder is that the minimum value of length l is larger than the corresponding value on the right side of the following expression

$$l > R\varepsilon^{-1} \tag{2}$$

Moreover, in general, Sanders' mid-long cylindrical shell theory is a first approximation theory, i.e., a theory that is consistent with the Kirchhoff hypotheses.

2.1 Complex cylindrical shell equation

Let the coordinates be at axial distance zR and circumferential angle θ. Dimensionless displacements u, v, and w, Goldenveizer stress functions χ_z, χ_θ, and ψ, stretching strains E_z, E_θ, and $E_{z\theta}$, bending strains K_z, K_θ, and $K_{z\theta}$, membrane stresses N_z, N_θ, and $N_{z\theta}$ and bending stresses M_z, M_θ, and $M_{z\theta}$ are given as follows

$$\begin{aligned}
&(\bar{u},\bar{v},\bar{w}) = \frac{\sigma R}{E}(u,v,\varepsilon^{-2}w)\\
&(\bar{\chi}_z,\bar{\chi}_\theta,\bar{\psi}) = \sigma R^2 h\varepsilon^2(\chi_z,\chi_\theta,\varepsilon^{-2}\psi)\\
&(\bar{E}_z,\bar{E}_\theta,\bar{E}_{z\theta}) = \frac{\sigma}{E}(E_z,E_\theta,E_{z\theta})\\
&(\bar{K}_z,\bar{K}_\theta,\bar{K}_{z\theta}) = \frac{\sigma}{ER\varepsilon^2}(K_z,K_\theta,K_{z\theta})\\
&(\bar{N}_z,\bar{N}_\theta,\bar{N}_{z\theta}) = \sigma h(N_z,N_\theta,N_{z\theta})\\
&(\bar{M}_z,\bar{M}_\theta,\bar{M}_{z\theta}) = \sigma Rh\varepsilon^2(M_z,M_\theta,M_{z\theta})
\end{aligned} \tag{3}$$

where $(\bar{\ })$ denotes dimensional quantity, and σ is a reference stress.

By exploiting the static-geometric analogy and the permissible variability in the constitutive relationships, Simmonds (1966) gave a unified expression for a cylindrical shell control equation for a complex-valued variable $\Omega = w + \mathrm{i}\psi$ as follows

$$\nabla^4\Omega + \ddot{\Omega} - \mathrm{i}\varepsilon^{-2}(1+\mathrm{i}\varepsilon^2 c)\Omega'' = 0 \tag{4}$$

where ∇^4 is the biharmonic operator in z, θ coordinate, $(\ )' \equiv \partial/\partial z$, $(\dot{\ }) \equiv \partial/\partial\theta$, and c is a real constant of order one O(1). Complex characteristic functions Φ and φ set by Sanders satisfy respectively Eq.(4) and $\Phi'' = \varepsilon^2\varphi$. Then complex displacements, stress functions, stretching and bending strains, membrane and bending stresses can be given in terms of Φ and φ by

$$\begin{aligned}
&\varepsilon^2 u = \ddot{\Phi}' - \varepsilon^2\nu\varphi' && \varepsilon^2\chi_z = -\mathrm{i}\ddot{\Phi}' - \mathrm{i}\varepsilon^2\nu\varphi'\\
&\varepsilon^2 v = -\dddot{\Phi} - \varepsilon^2(2+\nu)\dot{\varphi} && \varepsilon^2\chi_\theta = \mathrm{i}\dddot{\Phi} + \mathrm{i}\varepsilon^2(2-\nu)\dot{\varphi}\\
&w = -\ddot{\Phi} + \mathrm{i}(1+\mathrm{i}\varepsilon^2 c)\varphi && \psi = \mathrm{i}\ddot{\Phi} + (1+\mathrm{i}\varepsilon^2 c)\varphi\\
&E_z = \ddot{\varphi} - \nu\varphi'' && K_z = -\mathrm{i}\varphi''\\
&E_\theta = \varphi'' - \nu\ddot{\varphi} && K_\theta = -\mathrm{i}\ddot{\varphi}\\
&E_{z\theta} = -(1+\nu)\dot{\varphi}' && K_{z\theta} = -\mathrm{i}\dot{\varphi}'\\
&N_z = \ddot{\varphi} && M_z = -\mathrm{i}(\varphi'' + \nu\ddot{\varphi})\\
&N_\theta = \varphi'' && M_\theta = -\mathrm{i}(\ddot{\varphi} + \nu\varphi'')\\
&N_{z\theta} = -\dot{\varphi}' && M_{z\theta} = -\mathrm{i}(1-\nu)\dot{\varphi}'
\end{aligned} \tag{5}$$

where all physical quantities are the real parts. In this way, problem solving is at last turned to find Φ or φ to satisfy control Eq.(4) and the boundary conditions.

2.2 Equations corresponding to different states of stress

It is not easy to solve Eq.(4). Noting that there is a small parameter ε in the equation, it will be helpful to apply the singular perturbation theory. In general, variations of stress along the axial and circumferential directions in cylindrical shells are different. It depends on the geometrical characteristics of the cylindrical shell, external loading conditions, and boundary conditions. According to the small parameter ε and the features of different stress states, after taking power series expansion of the characteristic function, and then comparing the coefficients of ε in different power levels, the asymptotic functions of different orders under different stress states can finally be obtained. Due to the geometric feature of the cylindrical shell, the theoretical error magnitude of the first approximation is comparable to the intrinsic error of the Kirchhoff-Love thin shell theory, so that the second approximation usually will not be considered.

On the basis of how rapidly the physical quantity varies as the function of z and θ, as for cylindrical shell, there are three major cases as follows to be considered. In considering from Eq.(5), it is possible to directly discuss the variation of characteristic functions along the coordinate.

1) Suppose the characteristic function varies slowly in the z direction but not too rapidly in the θ direction. With defined scaled variables $\xi=\varepsilon z$ and $\eta=\theta$, the asymptotic expansion of the characteristic function will be

$$\Omega(\varepsilon^{-1}\xi,\eta)=\Omega_0(\xi,\eta)+\varepsilon^2\Omega_2(\xi,\eta)+\cdots \tag{6}$$

Substitute Eq.(6) into Eq.(4), and take the first approximation to get the governing equation of the semi-membrane theory

$$(\ddot{\Omega}+\Omega)^{\cdot\cdot}-\mathrm{i}\varepsilon^{-2}\Omega''=0 \tag{7}$$

If the first term in the asymptotic expansion of the characteristic function in the power of ε^2 is reserved in the asymptotic equation, then the more precise equation for the semi-membrane theory can be found as follows

$$(\ddot{\Omega}+\Omega)^{\cdot\cdot}+2\ddot{\Omega}''+(c-\mathrm{i}\varepsilon^{-2})\Omega''=\mathrm{O}(\varepsilon^4) \tag{8}$$

Under the semi-membrane state of stress, the displacements and stress functions in Eq.(5) can be expressed in simplified form as

$$\begin{array}{lll} \varepsilon^2 u=\ddot{\Phi}' & \varepsilon^2 v=-\dddot{\Phi} & w=-\ddot{\Phi}+\mathrm{i}\varphi \\ \varepsilon^2\chi_z=-\mathrm{i}\ddot{\Phi}' & \varepsilon^2\chi_\theta=\mathrm{i}\dddot{\Phi} & \psi=\mathrm{i}\ddot{\Phi}+\varphi \end{array} \tag{9}$$

2) Suppose the characteristic function varies rapidly in the z direction, but not too rapidly in the θ direction. Introducing scaled variables $\xi=\varepsilon^{-1}z$ and $\eta=\theta$, the governing equation of edge effect can be found as

$$\Omega''-\mathrm{i}\varepsilon^{-2}\Omega=0 \tag{10}$$

Under edge effect state of stress, the displacements and stress functions in Eq.(5) can be expressed as

$$\begin{array}{lll} u=-\nu\varphi' & v=-(2+\nu)\dot{\varphi} & w=\mathrm{i}(1+\mathrm{i}\varepsilon^2 c)\varphi \\ \chi_z=-\mathrm{i}\nu\varphi' & \chi_\theta=\mathrm{i}(2-\nu)\dot{\varphi} & \psi=(1+\mathrm{i}\varepsilon^2 c)\varphi \end{array} \tag{11}$$

3) Suppose the characteristic function varies rapidly with z and θ. Appropriate scaled variables are $\xi=\varepsilon^{-1}z$ and $\eta=\varepsilon^{-1}\theta$, and the shallow shell equation can be found as

$$\nabla^4\Omega-\mathrm{i}\varepsilon^{-2}\Omega''=0 \tag{12}$$

2.3 Solutions for mid-long cylindrical shell

Sanders pointed out that a complete solution of the mid-long cylindrical shell may be composed of the above mentioned semi-membrane solution, edge effect solution, shallow shell solution, and some other elementary solutions. Elementary solutions usually are composed of simple axial tension solutions, beam bending solutions, rigid-body motion solutions (for which stresses disappear), and null solutions (for which displacements disappear).

1) Since Φ and φ both satisfy the governing equation of the semi-membrane theory, the Fourier series solution to Eq.(7) can be given by

$$\Phi(z,\theta)=\sum_{-\infty}^{\infty}F_n\exp[\mathrm{i}^{3/2}|n|(n^2-1)^{1/2}\varepsilon z+\mathrm{i}n\theta] \tag{13}$$

where F_n represents the Fourier coefficients of $\Phi(0,\theta)$

$$F_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(0,\theta)\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta \quad (|n|=2,3,\cdots) \tag{14}$$

To be noted here is that $|n|$ begins from 2 to separate off F_0, F_{-1}, and F_1 from $\Phi(0,\theta)$. In other words, set $F_0=F_{-1}=F_1=0$. Mathematically, if F_0, F_{-1}, and F_1 are not separated off, since these coefficients do not contain a z term, and then from Eq.(13), when $z\to\infty$, Φ should converge to F_0, F_{-1} and F_1. In physics, concerning the mid-long cylindrical shell, the semi-membrane solution at a place sufficiently far away from the boundary should converge to the former elementary solutions. In other words, solutions corresponding to F_0, F_{-1}, and F_1 are elementary solutions. The reason to separate off F_0, F_{-1}, and F_1 is to make the solving process to be easy and convenient as the usual tension solution and bending solution for mid-long cylindrical shell exist already. It can be found from the above discussion that if the elementary solutions are given in another way, the necessary and sufficient condition for Eq.(13) and (14) will be

$$\int_{-\pi}^{\pi}\Phi(0,\theta)\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta=0 \qquad (|n|=0,\ 1) \tag{15}$$

Once a function, $\Phi(0,\theta)$ satisfies the Eq.(15); it is the solution to Eq.(7).

In Eq.(13), if $n^2-1/2$ is taken to replace $|n|\sqrt{n^2-1}$, a new series will be found. The error of the new series relative to the original one is less than 2% for n=2 and much less for larger values of n. From this new series, the following relationship can be derived

$$\Phi'=-\mathrm{i}^{3/2}\varepsilon(\ddot{\Phi}+\frac{1}{2}\Phi) \tag{16}$$

As for φ, a similar expression can be obtained

$$\varphi'=-\mathrm{i}^{3/2}\varepsilon(\ddot{\varphi}+\frac{1}{2}\varphi) \tag{17}$$

The relationship of $\Phi''=\varepsilon^2\varphi$ and Eq.(7) leads to

$$\varphi=-\mathrm{i}(\ddot{\Phi}+\Phi)^{\cdot\cdot} \tag{18}$$

Eq.(16) and Eq.(17) show that, for the characteristic functions, the derivative of the axial coordinate can be expressed by the derivative of the corresponding circumferential coordinate, while Eq.(18) shows that φ may also be expressed by the derivative of Φ relative to the circumferential coordinate. From Eq.(15) to Eq.(18), a series of expressions for characteristic functions of coordinates θ can be found, on which basis the boundary value solution of the equation will be able to be found.

2) From Eq.(10), it is not difficult to find the edge effect solution. On account of the relationship $\Phi'' = \varepsilon^2\varphi$, the following expressions for the solution can be found

$$\varphi = \mathrm{i}^{1/2} G(\theta) \exp[-\mathrm{i}^{1/2}\varepsilon^{-1} z] \tag{19}$$

$$\Phi = -\mathrm{i}^{3/2}\varepsilon^4 G(\theta) \exp[-\mathrm{i}^{1/2}\varepsilon^{-1} z] \tag{20}$$

It can be seen that Φ is $O(\varepsilon^4)$ and is very small and usually can be neglected.

3) As for the elementary solutions such as the simple axial tension solution, beam bending solution, rigid-body motion solution, and null solution, all have been discussed by Sanders (1982, 1987).

4) As for the mid-long cylindrical shell, the boundary data is 'smooth' or 'slowly varying' in most practical problems and the shallow shell solution may be neglected.

2.4 Boundary conditions of stress functions

From Eq.(5), the displacement, membrane and bending stresses can be expressed by Φ, φ, and their derivatives. As there is a static-geometric analogy between displacements and stress functions, if the boundary conditions on stress measures can be equivalently expressed in terms of conditions on stress functions, the treatment of the boundary condition will be simplified. Expression for the stress functions in terms of integrals of the effective Kirchhoff edge resultants was derived by Sanders (1980b). In the mid-long cylindrical shell as shown in Fig.1, expressions for the boundary values of the stress functions in terms of prescribed edge load T_z, T_θ, V and M_n acting on the edge z=0 are as follows

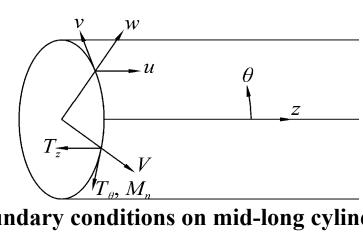

Fig.1 Boundary conditions on mid-long cylindrical shell

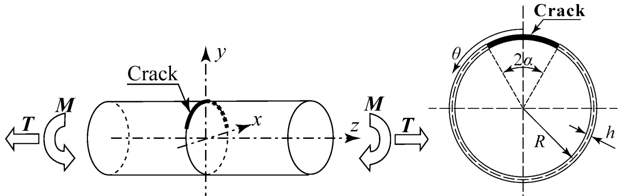

Fig.2 A cracked cylinder under loads

$$\begin{aligned}
\varepsilon^2\chi_z &= \sin\theta\int^\theta (T_\theta \sin\eta + \varepsilon^2 V\cos\eta)\mathrm{d}\eta + \\
&\quad \cos\theta\int^\theta (T_\theta\cos\eta - \varepsilon^2 V\sin\eta)\mathrm{d}\eta - \int^\theta T_\theta \mathrm{d}\eta \\
\varepsilon^2\chi_\theta &= \sin\theta\int^\theta (T_z + \varepsilon^2 M_n)\sin\eta\mathrm{d}\eta + \\
&\quad \cos\theta\int^\theta (T_z + \varepsilon^2 M_n)\cos\eta\mathrm{d}\eta - \int^\theta T_z\mathrm{d}\eta \\
\psi &= \sin\theta\int^\theta (T_z + \varepsilon^2 M_n)\cos\eta\mathrm{d}\eta + \\
&\quad \cos\theta\int^\theta (T_z + \varepsilon^2 M_n)\sin\eta\mathrm{d}\eta \\
\psi' &= -\sin\theta\int^\theta (T_\theta\sin\eta + \varepsilon^2 V\cos\eta)\mathrm{d}\eta - \\
&\quad \cos\theta\int^\theta (T_\theta\cos\eta - \varepsilon^2 V\sin\eta)\mathrm{d}\eta
\end{aligned} \tag{21}$$

3 Engineering application of Sanders' mid-long cylindrical shell theory

The basic contents of the Sanders' mid-long cylindrical shell theory have been introduced, and the derivation details may be found from references (Sanders, 1972, 1980a, 1983a). As for the mid-long cylindrical shell, the solution process is mainly focused on the semi-membrane and edge effect solution in most ocean engineering problems.

3.1 Mid-long cylindrical shell with circumferential crack

Due to the heavy cyclic wave loads, fatigue cracks, which are a common and dangerous kind of damage, may occur in the cylindrical shell acting as a supporting structure. In this engineering problem, physical quantities do not vary rapidly in the θ direction, except at a very small region near the crack tip. Based on Sanders' mid-long cylindrical shell theory, the linear elastic solution is given in Sanders (1982, 1983b). Here, only the treatment of boundary conditions and the magnitude analysis are given to show its simplicity and practicality.

In Fig.2, a mid-long cylinder with a circumferential crack subjected to combined tension T and bending M is shown. The cracked section is at z=0 and the opening angle of the crack is 2α. From the condition of symmetry, the cross section z=0 is the border. On the free surface of the crack there is no acting force, or T_θ=T_z=M_n=V=0. From Eq.(21), the boundary conditions may be written as

$$\chi_z = \chi_\theta = \psi = \psi' = 0 \qquad |\theta| < \alpha \tag{22}$$

Off the crack, symmetry consideration (T_θ=V=0) leads to

$$\chi_z = \psi' = 0\,,\quad u = w' = 0 \qquad \alpha < \theta < 2\pi - \alpha \tag{23}$$

The control Eq.(4) can be written as follows

$$\nabla^4\Omega + \ddot{\Omega} - \mathrm{i}\varepsilon^{-2}(1 + 2\mathrm{i}\varepsilon^2)\Omega'' = 0 \tag{24}$$

By means of Eq.(5), the boundary conditions can be restated in terms of Φ and φ

$$\begin{cases} \Re\{(1+\mathrm{i}\varepsilon^2\nu)\varphi_c\}=0 \\ \Re\{\mathrm{i}\ddot{\Phi}_c+\mathrm{i}\varepsilon^2(2-\nu)\varphi_c'\}=0 \\ \Re\{\varphi_c'+\mathrm{i}\varepsilon^2(2-\nu)\varphi_c'\}=0 \\ \Re\{\mathrm{i}\ddot{\Phi}_c'+\mathrm{i}\varepsilon^2\nu\varphi_c\}=0 \end{cases} \quad |\theta|<\alpha \tag{25}$$

$$\varphi_c'=0\,,\ \ \ddot{\Phi}_c'=0 \qquad \alpha<\theta<2\pi-\alpha \tag{26}$$

where $\Re\{\ \}$ denotes taking the real part of the variable in the brackets and the subscript c denotes the complete solution. The composition of the complete solution can be expressed as

$$\Phi_c=\Phi_b+\Phi \tag{27}$$

Here, Φ_b means the elementary solutions and Φ can be thought of as the solution incurred by the existence of a crack. As for the mid-long cylindrical shell, it can be easily understood that the influence of crack is limited along the z direction. That is to say, at a place sufficiently far away from the border, the complete solution should converge to corresponding elementary solutions such as the simple axis tension solution, beam bending solution, rigid body motion, and null solutions (solutions composed of certain complex constants to be specified). In other words, as to the solution incurred by the existence of a crack, when $z\to\infty$, there will be $\Phi\to 0$ or $\varphi\to 0$, and it is a linear combination of an edge effect solution and semi-membrane solution.

From Eq.(27), Eq.(25) and (26) can be written in the following forms

$$\begin{cases} \Re\{(1+\mathrm{i}\varepsilon^2\nu)(\varphi_b+\varphi)\}=0 \\ \Re\{\mathrm{i}(\ddot{\Phi}_b+\ddot{\Phi})+\mathrm{i}\varepsilon^2(2-\nu)(\varphi_b'+\varphi')\}=0 \\ \Re\{[1+\mathrm{i}\varepsilon^2(2-\nu)](\varphi_b'+\varphi')\}=0 \\ \Re\{\mathrm{i}(\ddot{\Phi}_b'+\ddot{\Phi}')+\mathrm{i}\varepsilon^2\nu(\varphi_b+\varphi)\}=0 \end{cases} \quad |\theta|<\alpha \tag{28}$$

$$\varphi_b'+\varphi'=0\,,\ \ \ddot{\Phi}_b'+\ddot{\Phi}'=0 \qquad \alpha<\theta<2\pi-\alpha \tag{29}$$

Taking a cracked cylindrical shell under axial tension as an example, the elementary solutions given in reference (Sanders, 1982) are substituted into the above equation and then the boundary conditions can be expressed by the solution with the existence of a crack as follows

$$\begin{cases} \Re\{(1+\mathrm{i}\varepsilon^2\nu)\varphi\}=-a_R+\varepsilon^2\nu a_I-\dfrac{1}{2}\theta^2 \\ \Re\{\mathrm{i}\ddot{\Phi}+\mathrm{i}\varepsilon^2(2-\nu)\varphi\}=a_R-\varepsilon^2\nu a_I+c_R\cos\theta-1+\dfrac{1}{2}\theta^2 \\ \Re\{\varphi'+\mathrm{i}\varepsilon^2(2-\nu)\varphi'\}=\varepsilon b_I+\varepsilon^3(2-\nu)b_R \\ \Re\{\mathrm{i}\ddot{\Phi}'+\mathrm{i}\varepsilon^2\nu\varphi'\}=-\varepsilon b_I-\varepsilon^3(2-\nu)b_R+\varepsilon d_I\cos\theta \end{cases} \quad |\theta|<\alpha \tag{30}$$

$$\begin{cases} \varphi'=-\mathrm{i}\varepsilon b \\ \ddot{\Phi}'=\varepsilon(1+2\mathrm{i}\varepsilon^2)b-\varepsilon d\cos\theta \end{cases} \quad \alpha<\theta<2\pi-\alpha \tag{31}$$

Here a, b, c, and d, which come from the elementary solutions, are complex constants to be determined and the subscripts I or R refer to the imaginary or real part of these constants.

The governing equations of semi-membrane and edge effects are shown as Eq.(7) and Eq.(10), respectively. Their solutions are given by Eq.(13), Eq.(19), and Eq.(20). In order to match the semi-membrane solution with an edge effect solution, it is necessary in the asymptotic equation to reserve the first term in the asymptotic expansion of the characteristic function in the power of ε^2. Considering that $\Phi''=\varepsilon^2\varphi$, Eq.(8) can be written as

$$(\ddot{\Phi}+\Phi)^{\cdot\cdot}-\mathrm{i}\varphi+2\varepsilon^2(\ddot{\varphi}+\varphi)=\mathrm{O}(\varepsilon^4) \tag{32}$$

It seems plausible that there should be no edge effect off the crack ($\alpha<\theta<2\pi-\alpha$). On the crack ($|\theta|<\alpha$), it is supposed that the matching form of the semi-membrane solution with an edge effect solution is $\varphi=\varphi_m+\varepsilon^2\varphi_e$, where φ_m denotes the semi-membrane solution and φ_e denotes the edge effect solution. From Eq.(20), it can be found that corresponding Φ_e is $\mathrm{O}(\varepsilon^4)$, which is too small to be taken into consideration. Considering that the variation of the semi-membrane solution and edge effect solution along the axial direction is different, the following expressions could be obtained

$$\varphi_m'=\mathrm{O}(\varepsilon\varphi_m)\,,\quad \Phi_m'=\mathrm{O}(\varepsilon\Phi_m)\,,\quad \varphi_e'=\mathrm{O}(\varepsilon^{-1}\varphi_e) \tag{33}$$

Taking the second derivative of θ in the second equation of Eq.(30), it becomes

$$\Re\{\mathrm{i}\ddot{\ddot{\Phi}}+\mathrm{i}\varepsilon^2(2-\nu)\ddot{\varphi}\}=-c_R\cos\theta+1 \tag{34}$$

With Eq.(34) and the first and second equations in Eq.(30), then

$$\Re\{\mathrm{i}(\ddot{\Phi}+\Phi)^{\cdot\cdot}+\mathrm{i}\varepsilon^2(2-\nu)\ddot{\varphi}+(1+2\mathrm{i}\varepsilon^2)\varphi\}=0 \tag{35}$$

Substitute $\varphi=\varphi_m+\varepsilon^2\varphi_e$ and $\Phi=\Phi_m$ into the above equation, and Eq.(35) is rewritten as

$$\begin{aligned} &\Re\{\mathrm{i}(\ddot{\Phi}_m+\Phi_m)^{\cdot\cdot}+\mathrm{i}\varepsilon^2(2-\nu)(\ddot{\varphi}_m+\varepsilon^2\ddot{\varphi}_e)+ \\ &(1+2\mathrm{i}\varepsilon^2)(\varphi_m+\varepsilon^2\varphi_e)\}=0 \end{aligned} \tag{36}$$

With Eq.(32), Eq.(36) turns into

$$\Re\{-\mathrm{i}\nu\ddot{\varphi}_m+\mathrm{i}\varepsilon^2(2-\nu)\ddot{\varphi}_e+(1+2\mathrm{i}\varepsilon^2)\varphi_e\}=0 \tag{37}$$

If high order items with small quantity ε^2 are neglected, then

$$\Re\{\varphi_e\}=\Re\{\mathrm{i}\nu\ddot{\varphi}_m\} \tag{38}$$

For the same reason, from the third and fourth equations in Eq.(30), together with Eq.(32) and (33), then

$$\Re\{\varphi_e'\}=0 \tag{39}$$

Substituting Eq.(19) into Eq.(38) and Eq.(39), it can be derived that G is real and

$$\Re\{\mathrm{i}^{1/2}G\}=\Re\{\mathrm{i}\,\nu\ddot{\varphi}_m(0,\theta)\} \tag{40}$$

From the above derivation it is found that edge effect solution may be determined by boundary values of the semi-membrane solution. Similarly, if the same matching form is taken, from the boundary condition $\alpha<\theta<2\pi-\alpha$ expressed in Eq.(31), the following equation can be obtained

$$\varphi_e'=0 \tag{41}$$

This shows that there will be no edge effect off the crack.

Eqs.(38), (39), (40), and (41) show that the solution incurred by the existence of the crack will be finally concluded to find a semi-membrane solution satisfying the boundary conditions Eqs.(30), (31), and governing Eq.(7). In view of Eq.(32), the first and third equations in the boundary condition Eq.(30) and the first equation in Eq.(31) are not independent of the second and fourth equations of Eq.(30) and the second equation of Eq.(31), respectively. Therefore in the range of the first approximation, the boundary conditions satisfied by the semi-membrane solution may be written as

$$\begin{cases}\Re\{\mathrm{i}\ddot{\Phi}_m\}=a_R+c_R\cos\theta-1+\dfrac{1}{2}\theta^2\\ \Re\{\mathrm{i}\ddot{\Phi}_m'\}=-\varepsilon b_I+\varepsilon d_I\cos\theta\end{cases}\quad |\theta|<\alpha \tag{42}$$

$$\ddot{\Phi}_m'=\varepsilon b-\varepsilon d\cos\theta \qquad \alpha<\theta<2\pi-\alpha \tag{43}$$

As for Φ_m' in Eqs.(42) and (43), and noting Eq.(16), it will be easy to obtain the ordinary differential equation of the second order. Several constants to be specified will be generated during the solving process. These constants together with the formerly mentioned a, b, c, and d become the unknowns in the boundary value solution system. On the boundary, physical quantities as displacement and stress functions should be continuous at place α, or the characteristic function $\ddot{\Phi}_m$ (in boundary conditions Eqs.(42) and (43)) itself and the several orders of derivation of θ should be continuous at the place α. Applying the above mentioned Eq.(15), a series of algebraic equations will be found. There will be no mathematical difficulty, and unknowns a, b, c, and d can be obtained from this algebraic equation system. The continuity condition of Eqs.(42) and (43) guarantees the solution to satisfy the boundary conditions, while the application of Eq.(15) means that the solution satisfies Eq.(7).

It should be noted that in the above solution, the semi-membrane solution and the edge effect solution utilize the matching form $\varphi=\varphi_m+\varepsilon^2\varphi_e$, because the boundary condition satisfied by the edge effect solution must be non-homogeneous and consistent. "Non-homogeneous" means that the semi-membrane solution should appear at least once in Eq.(38) or Eq.(39), otherwise there is only a null solution among the edge effect solutions; "consistent" means that the edge effect solution should appear in both Eq.(38) and Eq.(39), otherwise the number of boundary conditions solely satisfied by the semi-membrane solution will exceed two, but from the number of the order of Eq.(7), the solution process needs and can satisfy only two boundary conditions. Accordingly, if $\varphi=\varphi_m+\varphi_e$ is taken as a matching form, the derivation will result in G=0, which cannot satisfy the non-homogeneous condition. This shows that the magnitude order of the edge effect is overestimated. If the matching form is taken as $\varphi=\varphi_m+\varepsilon^4\varphi_e$, it will be found in the derivation that the number of boundary conditions solely satisfied by the semi-membrane solution will exceed two, which will not satisfy the requirement of consistency. This shows that the magnitude order of the edge effect is underestimated. In the linear elastic analysis for a circumferential crack at the fixed end of a mid-long cylinder (Alabi and Sanders, 1985), a similar conclusion can also be obtained. From references (Snaders, 1982; Alabi and Sanders, 1985), in considering a mid-long cylinder with a circumferential crack, the influence of the shallow shell solution and edge effect solution may be neglected in the range of the first approximation. The whole solving process now comes down to directly finding the semi-membrane solution, and this conclusion can largely reduce the difficulty of solving similar kinds of problems.

Sanders (1987) started directly from the semi-membrane theory associated with the Dugdale model and derived the elastic-plastic solution for a circumferential crack in the mid-long cylindrical shell loaded by bending. Based on a hypothesis that the crack opening shape in the neighborhood of the crack tip remains invariant as the crack grows, the elastic-plastic solution for the stable extension of the crack was given by Sanders (1987). In later papers, Zhao (1998, 1999) succeeded in extending the above results to the loading cases of axial tension and combined tension and bending. However, the crack cited in these papers was assumed to lie far away from any boundary or support. In fact, a cylindrical shell in a structure system is likely to be connected to other relatively stiff structures, and most cracks may initiate near these joints. In particular, for some new pattern offshore drilling platforms, the legs of the platform usually connect with each other by the pipes. As the legs of the platform are considerably stiffer than the pipes, the latter is assumed to be fixed on the former. Based on the above consideration and in connection with the Dugdale model and the CTOA (crack tip opening angle) fracture criterion, the linear elastic result of Aabi (1985) is extended to the elastic-plastic range to further perfect Sanders' theoretical system. Fig.3 was drawn according to the calculated results.

In the figure, $\sigma = M/(\pi R^2 h\sigma_F)$ is a dimensionless bending load, σ_F is the plastic flowing stress, α denotes the half angle of the crack, α_0 denotes the initial half angle of the crack, and ψ_c denotes the critical CTOA. The result in the figure clearly shows the variation of carrying capacity under different α_0 and ψ_c, and automatically traces the ultimate carrying capacity as the crack grows. Since the commercial general purpose finite element program at present has no module to analyze the elastic-plastic crack extension, this result is important.

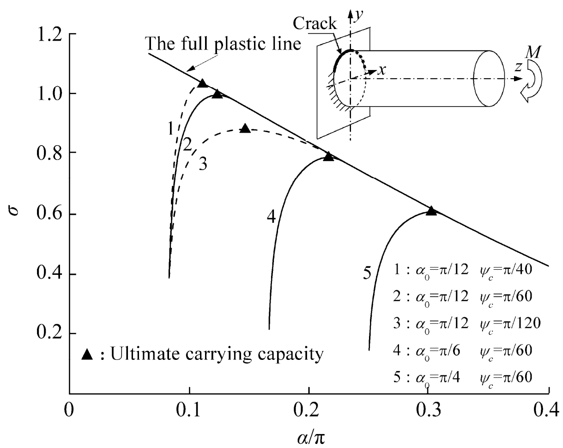

Fig.3 Relationship between load and crack extension

3.2 Error in Sanders' mid-long cylindrical shell theory

As for the mid-long cylindrical shell theory, the solution error comes mainly from the semi-membrane solution, which can be seen in reference (Zhao, 1999). It is worthy to point out that from the analysis of Sanders (1982), in the case of $D/h \leq 60$, the magnitude of error generated will not exceed that of the Kirchhoff-Love thin shell theory. The diameter-to-thickness ratio generally applied to supporting pipes of ocean platforms belongs to the above condition. Considering practical engineering application, Sanders' mid-long cylindrical shell theory gives sufficient precision. Moreover, the effectiveness of the semi-membrane solution will rely on the length of the cylindrical shell and Eq.(2) gives the suggested length. However, the study by Ye *et al*. (2008) stated that the above suggested length is somewhat conservative. On the other hand, the length suggested by Eq.(2) is still suitable for the supporting pipes in ocean engineering. In references (Tetsuya *et al*., 1989; Masahiko *et al*., 1991), as for the cylinder with crack damage, the values of CTOD (crack tip opening displacement) were calculated with the theory and contrasted with FEM results. It could be found that there is a good correlation between theoretical solutions and FEM results.

4 Conclusions

This paper introduced Sanders' mid-long cylindrical shell theory system along with detailed analysis and derivation. The major features of Sanders' mid-long cylindrical shell theory may be stated as follows. The physical characteristics of a mid-long cylindrical shell are sufficiently considered. Within the tolerence of error in the first approximation, ingnoring the influence of the shallow shell effect and edge effect, the solution can be obtained by the semi-membrane equation directly in certain engineering problems. In the solution process, with the static-geometric analogy between stress functions and displacements, the treatment of the boundary condition is simplified through the boundary conditions expressed by stress functions and the approximate relationships among derivatives of different coordinates.

Under the condition that there will be no special difficulty in mathematical derivation, the boundary value solution can be found by solving the algebraic equation set. The solution process is simple and clear, especially suitable to solve problems with complicated boundary conditions, and the closed form solution can usually be found. In practical engineering applications, it gives satisfactory precision.

This paper also provided some practical engineering application examples of Sanders' mid-long cylindrical shell theory. The calculation results given in this paper are for a mid-long cylinder with a circumferential crack at the fixed end under the consideration of crack extension. At present, it is still very difficult to obtain such results by general commercial finite element software.

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