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Green-Naghdi Theory, Part A: Green-Naghdi (GN) Equations for Shallow Water Waves

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Abstract: In this work, Green-Naghdi (GN) equations with general weight functions were derived in a simple way. A wave-absorbing beach was also considered in the general GN equations. A numerical solution for a level higher than 4 was not feasible in the past with the original GN equations. The GN equations for shallow water waves were simplified here, which make the application of high level (higher than 4) equations feasible. The linear dispersion relationships of the first seven levels were presented. The accuracy of dispersion relationships increased as the level increased. Level 7 GN equations are capable of simulating waves out to wave number times depth $kd < 26$. Numerical simulation of nonlinear water waves was performed by use of Level 5 and 7 GN equations, which will be presented in the next paper.
Keywords: Green-Naghdi (GN) equations; dispersion relation; wave-absorbing beach; shallow-water waves
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1 Introduction

The present work focuses on the simplification of Green-Naghdi (GN) equations. The linear dispersion relationships of the first seven levels of the GN equations are also obtained. The Green-Naghdi equations were originally developed by Green and Naghdi in 1974 to analyze some nonlinear free-surface flows. After the successful application of the method to nonlinear ship wave-making problems (Ertekin *et al*., 1986), the method was applied to many nonlinear water wave problems. Later, the model has was extended to deep-water waves by Webster and Kim (1990) and Xu *et al*. (1997) in two and three dimensions, respectively. Demirbilek and Webster (1992, 1999) applied the GN theory to shallow water and nonlinear wave propagation problems in two dimensions. Ertekin and Kim (1999) studied the hydroelastic behavior of a three-dimensional mat-type very large floating structures (VLFS) in oblique waves by use of the linear GN theory. Xia *et al*. (2008) studied the two-dimensional, nonlinear hydroelasticity of a mat-type very large floating structure (VLFS) within the scope of linear beam theory for structures and the nonlinear Level 1 GN theory for the fluids Zhao *et al*. (2011) applied the Level 2 GN equations to tsunami simulation.

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The original GN governing equations were not limited to irrotational flows. Kim *et al*. (2001) derived irrotational Green-Naghdi (IGN) equations. The deep-water version of the IGN equations has been derived and applied successfully to simulate ocean waves of extreme heights (Kim and Ertekin, 2000). The IGN equations were also derived for finite water-depth conditions, and tested numerically to show their self-convergence and accuracy (Kim *et al*., 2003).

Previously, the numerical solution for a level higher than 4 was not feasible with original GN equations. Zhao and Duan (2010) performed numerical calculations for the GN equations up to Level 3. They found that the results of Level 3 GN equations are better than that of Level 2 GN equations when compared with the experimental values. They did not perform calculations for a level higher than 4 because of the complexity of the original GN equations. As will be shown below, the numerical results of Level 5 GN equations are much better than Level 3 GN equations.

The simplification of the original GN equations is necessary to facilitate the application of the equations with a level higher than 4. Derivation of the original GN equations (Demirbilek and Webster, 1992) is complex and this method can be improved. In each numerical test, the self-convergence of the GN equations should be performed first. Then, the converged solutions can be compared with known experimental values and/or other solutions.

This paper is organized as follows. In Section 2 the GN equations with a wave-absorbing beach are derived in a general case. Section 3 provides the GN equations for shallow water waves, which are much simpler than the original GN equations. Section 4 provides the linear dispersion relationships of the first seven levels of GN equations for shallow water waves. Some conclusions are given in Section 5.

2 GN equations with general weight functions

The free-surface flow of an inviscid and incompressible fluid in water of variable depth is considered. The coordinate system is chosen such that the z-axis directs against gravity and the Oxy-plane is the still water level. The location of the free surface is denoted by $z=\beta(x,y,t)$ and the bottom as $z=\alpha(x,y,t)$. It should be noted that the bottom is allowed to vary with time in the GN model.

The conservation equations of momentum of the original GN model (Demirbilek and Webster, 1992) start from

$$\rho\vec{v}_{,t}+\left(\rho v_i\vec{v}\right)_{,i}=-p_{,i}\vec{e}_i-\rho g\vec{e}_3 \tag{1}$$

Here we derive the conservation equations of momentum of the GN model from

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\mu u=-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2a}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\mu v=-\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2b}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\mu w=-\frac{1}{\rho}\left(\frac{\partial p}{\partial z}+\rho g\right) \tag{2c}$$

where μ, a given function of (x,y), is the artificial damping coefficient. The terms μu, μv, and μw represent the wave-absorbing beach. Xu *et al*. (2007) introduced these terms into the GN equations for deep water waves. Although Eq.(2) and Eq.(1) are the same in essence, Eq.(1) makes the derivation of GN equations harder. The other conditions which the GN model should observe are:

Continuity condition:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{3}$$

Kinematic free surface and bottom condition:

$$w-\frac{\partial \beta}{\partial t}-u\frac{\partial \beta}{\partial x}-v\frac{\partial \beta}{\partial y}=0 \quad (z=\beta) \tag{4}$$

$$w-\frac{\partial \alpha}{\partial t}-u\frac{\partial \alpha}{\partial x}-v\frac{\partial \alpha}{\partial y}=0 \quad (z=\alpha) \tag{5}$$

Dynamic free surface and bottom condition:

$$p\big|_{z=\beta}=\hat{p}(x,y,t) \tag{6}$$

$$p\big|_{z=\alpha}=\bar{p}(x,y,t) \tag{7}$$

In the GN model, the velocity field is expressed approximately as

$$\begin{cases} u(x,y,z,t)=\sum_{n=0}^{K}u_n(x,y,t)\lambda_n(z) \\ v(x,y,z,t)=\sum_{n=0}^{K}v_n(x,y,t)\lambda_n(z) \\ w(x,y,z,t)=\sum_{n=0}^{K}w_n(x,y,t)\lambda_n(z) \end{cases} \tag{8}$$

where $\lambda_n(z)$ are called shape functions. The shape functions should accurately represent the vertical structure of the flow field being dealt with. For shallow water problems, 1, z, z^2, $\ldots$, can be chosen as shape functions. For deep water waves, e^{az}, $e^{az}z$, $e^{az}z^2$, $\ldots$,can be chosen as shape functions. The parameter a is a representative wave number. The coefficients u_n, v_n and w_n are unknown functions. K is called the level of the Green-Naghdi theory.

By use of Eq.(8), the equations of GN model can easily be obtained. They are:

Continuity condition:

$$\sum_{n=0}^{K}\left(\frac{\partial u_n}{\partial x}+\frac{\partial v_n}{\partial y}\right)\lambda_n(z)+\sum_{n=0}^{K}w_n\frac{\mathrm{d}\lambda_n(z)}{\mathrm{d}z}=0 \tag{9}$$

Kinematic free surface and bottom condition:

$$\frac{\partial \beta}{\partial t}=\sum_{n=0}^{K}\lambda_n(\beta)\left(w_n-\frac{\partial \beta}{\partial x}u_n-\frac{\partial \beta}{\partial y}v_n\right) \tag{10}$$

$$\frac{\partial \alpha}{\partial t}=\sum_{n=0}^{K}\lambda_n(\alpha)\left(w_n-\frac{\partial \alpha}{\partial x}u_n-\frac{\partial \alpha}{\partial y}v_n\right) \tag{11}$$

A weak formulation similar to the Galerkin approach is employed where the shape functions λ_n are used as weighting functions to develop K approximate equations which express the conservation of momentum in some integral sense.

Taking Eq.(2a) for example, the following can easily be gotten:

$$\sum_{m=0}^{K}\left[\frac{\partial u_m}{\partial t}\lambda_m+\sum_{r=0}^{K}\left(\frac{\partial u_m}{\partial x}u_r+\frac{\partial u_m}{\partial y}v_r\right)\lambda_m\lambda_r+\sum_{r=0}^{K}u_m w_r\frac{\mathrm{d}\lambda_m}{\mathrm{d}z}\lambda_r+\mu u_m\lambda_m\right]= -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{12}$$

Multiply (12) by each λ_n and integrating through the vertical direction results from bottom α to free surface β. The result is:

$$\sum_{m=0}^{K}\left[\frac{\partial u_m}{\partial t}S_{mn}+\sum_{r=0}^{K}\left(\frac{\partial u_m}{\partial x}u_r+\frac{\partial u_m}{\partial y}v_r\right)S_{mrn}+\sum_{r=0}^{K}u_m w_r S_{rn}^{m}+\mu u_m S_{mn}\right]= \frac{1}{\rho}\left[-\frac{\partial P_n}{\partial x}+\hat{p}\lambda_n(\beta)\frac{\partial \beta}{\partial x}-\bar{p}\lambda_n(\alpha)\frac{\partial \alpha}{\partial x}\right] \tag{13}$$

where

$S_{mn}=\int_\alpha^\beta \lambda_m\lambda_n \mathrm{d}z$, $S_{mrn}=\int_\alpha^\beta \lambda_m\lambda_r\lambda_n \mathrm{d}z$, $S_{rn}^{m}=\int_\alpha^\beta \frac{d\lambda_m}{dz}\lambda_r\lambda_n \mathrm{d}z$,

$P_n=\int_\alpha^\beta p\lambda_n \mathrm{d}z$.

Similarly, the other two conservation equations of momentum can be obtained from Eqs.(2b) and (2c) easily. They are

$$\sum_{m=0}^{K}\left[\frac{\partial v_m}{\partial t}S_{mn}+\sum_{r=0}^{K}\left(\frac{\partial v_m}{\partial x}u_r+\frac{\partial v_m}{\partial y}v_r\right)S_{mrn}+\sum_{r=0}^{K}v_m w_r S_{rn}^{m}+\mu v_m S_{mn}\right]= \frac{1}{\rho}\left[-\frac{\partial P_n}{\partial y}+\hat{p}\lambda_n(\beta)\frac{\partial \beta}{\partial y}-\bar{p}\lambda_n(\alpha)\frac{\partial \alpha}{\partial y}\right] \tag{14}$$

$$\sum_{m=0}^{K}\left[\frac{\partial w_m}{\partial t}S_{mn}+\sum_{r=0}^{K}\left(\frac{\partial w_m}{\partial x}u_r+\frac{\partial w_m}{\partial y}v_r\right)S_{mrn}+\sum_{r=0}^{K}w_m w_r S_{rn}^{m}+\mu w_m S_{mn}\right]= \frac{1}{\rho}\left[P_n^{*}-\rho g S_n-\hat{p}\lambda_n(\beta)+\bar{p}\lambda_n(\alpha)\right] \tag{15}$$

for $n=0,1,2,\ldots,K$. where $S_n=\int_\alpha^\beta \lambda_n \mathrm{d}z$, $P_n^{*}=\int_\alpha^\beta p\frac{\mathrm{d}\lambda_n}{\mathrm{d}z}\mathrm{d}z$. Eqs.(6), (7), (9), (10), (11), (13), (14), and (15) form the GN equations with general weight functions.

3 GN equations for shallow water waves

For shallow water problems, the polynomial weighting function set is chosen as:

$$\lambda_n(z)=z^n \tag{16}$$

The velocity field is given by

$$\begin{cases} u(x,y,z,t)=u_0+u_1 z+u_2 z^2+\ldots+u_K z^K \\ v(x,y,z,t)=v_0+v_1 z+v_2 z^2+\ldots+v_K z^K \\ w(x,y,z,t)=w_0+w_1 z+w_2 z^2+\ldots+w_K z^K \end{cases} \tag{17}$$

The equations for kinematic boundary conditions, conservation of mass, and momentum can now be reduced using Eq.(16). The kinematic boundary conditions becomes

$$\frac{\partial \beta}{\partial t}=\sum_{n=0}^{K}\beta^n\left(w_n-\frac{\partial \beta}{\partial x}u_n-\frac{\partial \beta}{\partial y}v_n\right) \tag{18}$$

$$\frac{\partial \alpha}{\partial t}=\sum_{n=0}^{K}\alpha^n\left(w_n-\frac{\partial \alpha}{\partial x}u_n-\frac{\partial \alpha}{\partial y}v_n\right) \tag{19}$$

The continuity equation becomes

$$\left(\frac{\partial u_K}{\partial x}+\frac{\partial v_K}{\partial y}\right)z^K+\sum_{n=0}^{K-1}\left(\frac{\partial u_n}{\partial x}+\frac{\partial v_n}{\partial y}+w_{n+1}(n+1)\right)z^n=0 \tag{20}$$

If Eq.(20) is to be held everywhere, each coefficient of z^n must be set to zero, that is

$$\frac{\partial u_K}{\partial x}+\frac{\partial v_K}{\partial y}=0 \tag{21}$$

$$\frac{\partial u_n}{\partial x}+\frac{\partial v_n}{\partial y}+(n+1)w_{n+1}=0 \quad (n=0,1,\ldots,K-1) \tag{22}$$

Demirbilek and Webster (1992) introduced the restricted GN theory. They restricted the last component of the director. That is

$$u_K=0 \tag{23}$$

$$v_K=0 \tag{24}$$

The conditions for conservation of momentum become

$$E_n=\frac{1}{\rho}\left[-\frac{\partial P_n}{\partial x}+\hat{p}\beta^n\frac{\partial \beta}{\partial x}-\bar{p}\alpha^n\frac{\partial \alpha}{\partial x}\right] \tag{25}$$

$$F_n=\frac{1}{\rho}\left[-\frac{\partial P_n}{\partial y}+\hat{p}\beta^n\frac{\partial \beta}{\partial y}-\bar{p}\alpha^n\frac{\partial \alpha}{\partial y}\right] \tag{26}$$

$$G_n=\frac{1}{\rho}\left[P_n^{*}-\rho g S_n-\hat{p}\beta^n+\bar{p}\alpha^n\right] \tag{27}$$

For $n=0,1,2,\ldots,K$. Where

$$E_n=\sum_{m=0}^{K}\left[\frac{\partial u_m}{\partial t}S_{mn}+\sum_{r=0}^{K}\left(\frac{\partial u_m}{\partial x}u_r+\frac{\partial u_m}{\partial y}v_r\right)S_{mrn}+\sum_{r=0}^{K}u_m w_r S_{rn}^{m}+\mu u_m S_{mn}\right]$$

$$F_n=\sum_{m=0}^{K}\left[\frac{\partial v_m}{\partial t}S_{mn}+\sum_{r=0}^{K}\left(\frac{\partial v_m}{\partial x}u_r+\frac{\partial v_m}{\partial y}v_r\right)S_{mrn}+\sum_{r=0}^{K}v_m w_r S_{rn}^{m}+\mu v_m S_{mn}\right]$$

$$G_n=\sum_{m=0}^{K}\left[\frac{\partial w_m}{\partial t}S_{mn}+\sum_{r=0}^{K}\left(\frac{\partial w_m}{\partial x}u_r+\frac{\partial w_m}{\partial y}v_r\right)S_{mrn}+\sum_{r=0}^{K}w_m w_r S_{rn}^{m}+\mu w_m S_{mn}\right]$$

$S_n=\int_\alpha^\beta z^n \mathrm{d}z$, $S_{mn}=\int_\alpha^\beta z^{m+n}\mathrm{d}z$, $S_{mrn}=\int_\alpha^\beta z^{m+r+n}\mathrm{d}z$, $S_{rn}^{m}=m\int_\alpha^\beta z^{m+r+n-1}\mathrm{d}z$

Notice that

$$P_n^{*}=\int_\alpha^\beta p\frac{\mathrm{d}\lambda_n}{\mathrm{d}z}\mathrm{d}z=n\int_\alpha^\beta p z^{n-1}\mathrm{d}z=nP_{n-1} \tag{28}$$

With $n=0$, Eq.(27) becomes

$$G_0=\frac{1}{\rho}\left(-\rho g S_0-\hat{p}+\bar{p}\right) \tag{29}$$

One can easily get the pressure on the bottom from Eq.(29); it can be expressed by

$$\bar{p}=\rho G_0+\rho g S_0+\hat{p} \tag{30}$$

Eqs.(25) and (26) can be rewritten as

$$\frac{\partial P_n}{\partial x}=-\rho E_n+\hat{p}\beta^n\frac{\partial \beta}{\partial x}-\bar{p}\alpha^n\frac{\partial \alpha}{\partial x} \tag{31}$$

$$\frac{\partial P_n}{\partial y}=-\rho F_n+\hat{p}\beta^n\frac{\partial \beta}{\partial y}-\bar{p}\alpha^n\frac{\partial \alpha}{\partial y} \tag{32}$$

By using of Eq.(28), one can rewrite Eq.(27) as

$$P_n = \frac{1}{(n+1)}\left(\rho G_{n+1} + \rho g S_{n+1} + \hat{p}\beta^{n+1} - \overline{p}\alpha^{n+1}\right) \tag{33}$$

$\partial P_n / \partial x$ and $\partial P_n / \partial y$ are eliminated from Eqs.(31), (32),and (33). The following equations are obtained:

$$\frac{\partial}{\partial x}\left(\rho G_{n+1} + \rho g S_{n+1} + \hat{p}\beta^{n+1} - \overline{p}\alpha^{n+1}\right) - (n+1)\left(-\rho E_n + \hat{p}\beta^n \frac{\partial \beta}{\partial x} - \overline{p}\alpha^n \frac{\partial \alpha}{\partial x}\right) = 0 \tag{34}$$

$$\frac{\partial}{\partial y}\left(\rho G_{n+1} + \rho g S_{n+1} + \hat{p}\beta^{n+1} - \overline{p}\alpha^{n+1}\right) - (n+1)\left(-\rho F_n + \hat{p}\beta^n \frac{\partial \beta}{\partial y} - \overline{p}\alpha^n \frac{\partial \alpha}{\partial y}\right) = 0 \tag{35}$$

Eqs.(34) and (35) can be simplified to

$$\frac{\partial}{\partial x}\left(G_{n+1} + g S_{n+1}\right) + (n+1) E_n + \beta^{n+1}\frac{\partial}{\partial x}\left(\frac{\hat{p}}{\rho}\right) - \alpha^{n+1}\frac{\partial}{\partial x}\left(\frac{\overline{p}}{\rho}\right) = 0 \tag{36}$$

$$\frac{\partial}{\partial y}\left(G_{n+1} + g S_{n+1}\right) + (n+1) F_n + \beta^{n+1}\frac{\partial}{\partial y}\left(\frac{\hat{p}}{\rho}\right) - \alpha^{n+1}\frac{\partial}{\partial y}\left(\frac{\overline{p}}{\rho}\right) = 0 \tag{37}$$

for $n = 0,1,2,\ldots$.

Finally, the GN equations for shallow water waves can be gotten. They are

$$\frac{\partial \beta}{\partial t} = \sum_{n=0}^{K} \beta^n \left(w_n - \frac{\partial \beta}{\partial x} u_n - \frac{\partial \beta}{\partial y} v_n \right) \tag{38}$$

$$\frac{\partial}{\partial x}(G_n + gS_n) + nE_{n-1} - \alpha^n \frac{\partial}{\partial x}(G_0 + gS_0) + \left(\beta^n - \alpha^n\right)\frac{\partial}{\partial x}\left(\frac{\hat{p}}{\rho}\right) = 0 \tag{39}$$

$$\frac{\partial}{\partial y}(G_n + gS_n) + nF_{n-1} - \alpha^n \frac{\partial}{\partial y}(G_0 + gS_0) + \left(\beta^n - \alpha^n\right)\frac{\partial}{\partial y}\left(\frac{\hat{p}}{\rho}\right) = 0 \tag{40}$$

for $n = 1,2,3,\ldots,K$. Where

$$E_n = \sum_{m=0}^{K}\left[\frac{\partial u_m}{\partial t} S_{mn} + \sum_{r=0}^{K}\left(\frac{\partial u_m}{\partial x} u_r + \frac{\partial u_m}{\partial y} v_r\right) S_{mrn} + \sum_{r=0}^{K} u_m w_r S_{rn}^m + \mu u_m S_{mn}\right]$$

$$F_n = \sum_{m=0}^{K}\left[\frac{\partial v_m}{\partial t} S_{mn} + \sum_{r=0}^{K}\left(\frac{\partial v_m}{\partial x} u_r + \frac{\partial v_m}{\partial y} v_r\right) S_{mrn} + \sum_{r=0}^{K} v_m w_r S_{rn}^m + \mu v_m S_{mn}\right]$$

$$G_n = \sum_{m=0}^{K}\left[\frac{\partial w_m}{\partial t} S_{mn} + \sum_{r=0}^{K}\left(\frac{\partial w_m}{\partial x} u_r + \frac{\partial w_m}{\partial y} v_r\right) S_{mrn} + \sum_{r=0}^{K} w_m w_r S_{rn}^m + \mu w_m S_{mn}\right]$$

$$S_n = \int_\alpha^\beta z^n \mathrm{d}z\,,\ S_{mn} = \int_\alpha^\beta z^{m+n}\mathrm{d}z\,,\ S_{mrn} = \int_\alpha^\beta z^{m+r+n}\mathrm{d}z\,,\ S_{rn}^m = m\int_\alpha^\beta z^{m+r+n-1}\mathrm{d}z$$

$u_K = 0$, $v_K = 0$

$$w_n = -\frac{1}{n}\left(\frac{\partial u_{n-1}}{\partial x} + \frac{\partial v_{n-1}}{\partial y}\right) \quad \text{for} \quad n = 1,2,\ldots,K$$

$$w_0 = \frac{\partial \alpha}{\partial t} - \sum_{n=1}^{K}\alpha^n\left(w_n - \frac{\partial \alpha}{\partial x}u_n - \frac{\partial \alpha}{\partial y}v_n\right) + \frac{\partial \alpha}{\partial x}u_0 + \frac{\partial \alpha}{\partial y}v_0$$

Usually, the pressure is set at free surface $\hat{p} = 0$. For each choice of K , a complete closed set of equations is developed that is independent from those for a different value of K . Thus, the kinematic models form a hierarchy depending on K and increasing in complexity with K . Obviously, this hierarchy is different from a perturbation expansion. A terminology is adopted that describes the complexity of the theory, and henceforth, a particular member of these hierarchies is referred to as the "*K*th level approximation". The GN equations with K=1, K=2,… are called Level 1, Level 2,… GN equations, respectively.

4 Linear dispersion relationships of GN equations

The dispersion relationships for a small amplitude linear sinusoidal wave can be obtained from the linearized forms of Eqs.(38) and (39) in two dimensions.

Take Level 1 GN equations for example, set $\hat{p} = 0$, $\mu = 0$, and $\alpha = -d$. Water depth d is a constant. The linearized forms of Level 1 GN equations are

$$\frac{\partial \beta}{\partial t} + d\frac{\partial u_0}{\partial x} = 0 \tag{41}$$

$$3\frac{\partial u_0}{\partial t} + 3g\frac{\partial \beta}{\partial x} - d^2\frac{\partial^3 u_0}{\partial x^2 \partial t} = 0 \tag{42}$$

It is assumed that the free surface $\beta(x,t)$ and coefficient $u_0(x,t)$ can be expressed as

$$\beta(x,t) = A\cos(kx - \omega t) \tag{43}$$

$$u_0(x,t) = \tilde{u}_0 \cos(kx - \omega t) \tag{44}$$

Insert Eqs.(43) and (44) into Eq.(42); it is found that:

$$\tilde{u}_0 = \frac{3gA}{c(3 + d^2k^2)} \tag{45}$$

Insert Eqs.(43), (44),and (45) into Eq.(41), and the linear dispersion relationships of Level 1 GN equations can be gotten. They are:

$$c^2 = \frac{3gd}{3 + d^2k^2} \tag{46}$$

The analysis method of other levels of GN equations is similar. Here the non-dimensional dispersion relationships of the first seven levels of GN equations are given. They are

$$K = 1: \qquad \overline{c}^2 = \frac{3\overline{d}}{3 + \overline{d}^2} = \overline{d} \underline{- \frac{\overline{d}^3}{3}} + \frac{\overline{d}^5}{9} + O(\overline{d})^7 \tag{47}$$

$$K=2:\bar{c}^2=\frac{24\bar{d}\left(10+\bar{d}^2\right)}{240+104\bar{d}^2+3\bar{d}^4}=\underline{\bar{d}-\frac{\bar{d}^3}{3}}+\frac{19\bar{d}^5}{144}+O(\bar{d})^7 \quad (48)$$

$$K=3: \quad \bar{c}^2=\frac{15\bar{d}\left(420+52\bar{d}^2+\bar{d}^4\right)}{6300+2880\bar{d}^2+135\bar{d}^4+\bar{d}^6}$$

$$=\underline{\bar{d}-\frac{\bar{d}^3}{3}+\frac{2\bar{d}^5}{15}-\frac{17\bar{d}^7}{315}}+\frac{2893\bar{d}^9}{132300}+O(\bar{d})^{11} \quad (49)$$

$$K=4:\bar{c}^2=\frac{120\bar{d}\left(42336+5712\bar{d}^2+154\bar{d}^4+\bar{d}^6\right)}{5080320+2378880\bar{d}^2+134064\bar{d}^4+1800\bar{d}^6+5\bar{d}^8}$$

$$=\underline{\bar{d}-\frac{\bar{d}^3}{3}+\frac{2\bar{d}^5}{15}-\frac{17\bar{d}^7}{315}}+\frac{185173\bar{d}^9}{8467200}+O(\bar{d})^{11} \quad (50)$$

$K=5$:

$$\bar{c}^2=\underline{\bar{d}-\frac{\bar{d}^3}{3}+\frac{2\bar{d}^5}{15}-\frac{17\bar{d}^7}{315}+\frac{62\bar{d}^9}{2835}-\frac{1382\bar{d}^{11}}{155925}}+\frac{1324889\bar{d}^{13}}{368831232}+O(\bar{d})^{15} \quad (51)$$

$K=6$:

$$\bar{c}^2=\underline{\bar{d}-\frac{\bar{d}^3}{3}+\frac{2\bar{d}^5}{15}-\frac{17\bar{d}^7}{315}+\frac{62\bar{d}^9}{2835}-\frac{1382\bar{d}^{11}}{155925}}+\frac{1589866811\bar{d}^{13}}{442597478400}+O(\bar{d})^{15} \quad (52)$$

$K=7$:

$$\bar{c}^2=\underline{\bar{d}-\frac{\bar{d}^3}{3}+\frac{2\bar{d}^5}{15}-\frac{17\bar{d}^7}{315}+\frac{62\bar{d}^9}{2835}-\frac{1382\bar{d}^{11}}{155925}+\frac{21844\bar{d}^{13}}{6081075}-\frac{929569\bar{d}^{15}}{638512875}}+\frac{620626600207\bar{d}^{17}}{1051860569760000}+O(\bar{d})^{19} \quad (53)$$

where the non-dimensionalization is performed by $\bar{d}=kd$, $\bar{c}=c\sqrt{k/g}$. Only the dispersion relationships of the first four levels of GN equations are shown here. The Taylor series of the approximate dispersion relationships of the first seven levels are given to show their accuracy, and the underlined terms in the Taylor series are the terms equal to those of the exact value, which is

$$\bar{c}_{\text{exact}}{}^2=\tanh(\bar{d}) \quad (54)$$

The Taylor series of Eq.(54) is

$$\bar{c}_{\text{exact}}{}^2=\bar{d}-\frac{\bar{d}^3}{3}+\frac{2\bar{d}^5}{15}-\frac{17\bar{d}^7}{315}+\frac{62\bar{d}^9}{2835}-\frac{1382\bar{d}^{11}}{155925}+\frac{21844\bar{d}^{13}}{6081075}-\frac{929569\bar{d}^{15}}{638512875}+\frac{6404582\bar{d}^{17}}{10854718875}+O(\bar{d})^{19} \quad (55)$$

In Fig.1, the dispersion relationships of the first seven levels of the GN equations are compared with the exact value.

The accuracy of the dispersion relationships of the GN equations increases as the level of the equations increases. With the choice of $K=7$, the relative errors of $\left(c^2-c_{\text{exact}}^2\right)/c_{\text{exact}}^2$ are less than 2% for $kd<26$.

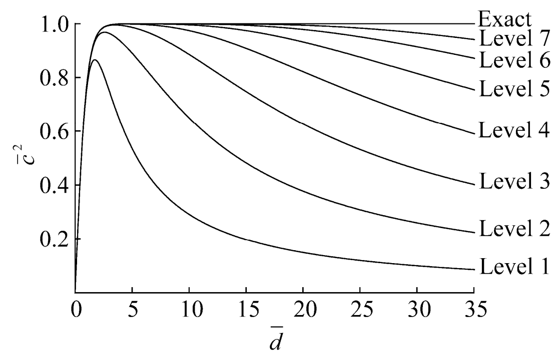

Fig.1 Dispersion relationships of the GN equations

The high order Boussinesq equation (Madsen *et al*., 2006; Fuhrman and Madsen, 2009) is one of the best Boussinesq equations. The dispersion relationships of the high order Boussinesq model are

$$\frac{\omega^2}{gdk^2}=\frac{1+\kappa^2\mu_2+\kappa^4\mu_4+\kappa^6\mu_6+\kappa^8\mu_8}{1+\kappa^2\nu_2+\kappa^4\nu_4+\kappa^6\nu_6+\kappa^8\nu_8+\kappa^{10}\nu_{10}} \quad (56)$$

where

$$\kappa=kd\ ,\quad \sigma=-\frac{\hat{z}}{d},\quad \mu_2=\frac{1}{6}-\frac{\sigma^2}{9}\ ,\quad \mu_4=\frac{1}{120}-\frac{\sigma^2}{54}+\frac{4\sigma^4}{567}$$

$$\mu_6=\frac{\sigma^2}{270}-\frac{\sigma^3}{72}+\frac{29\sigma^4}{1701}-\frac{\sigma^5}{135}+\frac{2\sigma^6}{2835}$$

$$\mu_8=\frac{\sigma^4}{7560}-\frac{\sigma^5}{1620}+\frac{17\sigma^6}{17010}-\frac{11\sigma^7}{17010}+\frac{8\sigma^8}{59535}\ ,\quad \nu_2=\frac{1}{2}-\frac{\sigma^2}{9}$$

$$\nu_4=\frac{1}{24}-\frac{\sigma^2}{18}+\frac{4\sigma^4}{567}\ ,\quad \nu_6=\frac{\sigma}{120}-\frac{5\sigma^2}{216}+\frac{\sigma^3}{54}-\frac{5\sigma^4}{1134}+\frac{\sigma^5}{945}-\frac{\sigma^6}{2835}$$

$$\nu_8=\frac{\sigma^3}{1080}-\frac{\sigma^4}{252}+\frac{7\sigma^5}{1215}-\frac{\sigma^6}{315}+\frac{4\sigma^7}{8505}+\frac{\sigma^8}{59535}$$

$$\nu_{10}=\frac{\sigma^5}{113400}-\frac{\sigma^6}{22680}+\frac{2\sigma^7}{25515}-\frac{\sigma^8}{17010}+\frac{\sigma^9}{59535}-\frac{\sigma^{10}}{893025}$$

The choice of $\sigma=0.5$ leads to an accuracy, which is very similar to the Padé(8,10) expansion, and less than 2% errors are found for $kd<25.7$.

The non-dimensional dispersion relationships of the high order Boussinesq equations (Madsen *et al*., 2006) can be written as:

$$\bar{c}^2=\frac{\bar{d}(1+\bar{d}^2\mu_2+\bar{d}^4\mu_4+\bar{d}^6\mu_6+\bar{d}^8\mu_8)}{1+\bar{d}^2\nu_2+\bar{d}^4\nu_4+\bar{d}^6\nu_6+\bar{d}^8\nu_8+\bar{d}^{10}\nu_{10}} \quad (57)$$

where $\bar{d}=\kappa=kd$. In Fig. 2, the dispersion relationships of the Level 7 GN equations are compared with the exact value and high order Boussinesq equations.

It should be noted that there are fourth and fifth-derivatives in the high order Boussinesq equations. In contrast, the highest order derivatives in GN equations are third-derivatives no matter how high the level is.

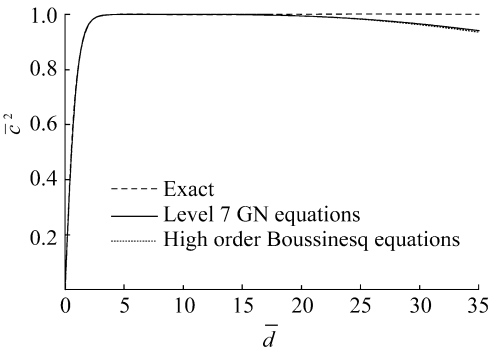

Fig.2 Comparison of dispersion relationships between Level 7 GN equations and high order Boussinesq equations (Madsen *et al*., 2006)

5 Conclusions

In this paper, the GN equations with general weight functions were derived in a simple way. A wave-absorbing beach was also considered in the general GN equations. The numerical solution for a level higher than 4 was not feasible in the past with the original GN equations. After the simplification of GN equations for shallow water waves, application of high level (higher than 4) equations was feasible. Numerical calculations has been performed by use of the simplified Level 5 and Level 7 GN equations, which will be presented in the next paper. The linear dispersion relationships of the first seven levels of GN equations were also presented here. The accuracy of dispersion relationships increased as the level increased. From Eqs.(47)–(53), it is found that Level 2, 4, and 6 did not improve much in their dispersion relationships compared with previous levels The Level 1, 2, 3, 4, 5, 6, and 7 GN equations have $O(\bar{d})^5$, $O(\bar{d})^5$, $O(\bar{d})^9$, $O(\bar{d})^9$, $O(\bar{d})^{13}$, $O(\bar{d})^{13}$, and $O(\bar{d})^{17}$ error on dispersion relationship compared with the exact value, respectively. In each numerical test, the self-convergence of the GN equations should be performed first by use of Level 1, 3, 5, and 7 GN equations. Then, the converged solutions can be compared with known experimental values and/or other solutions. Numerical simulation of nonlinear water waves will be presented in the next paper.

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