

# Effect of Bottom Undulation on the Waves Generated Due to Rolling of a Plate

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**Abstract:** In the present paper, the effect of a small bottom undulation of the sea bed in the form of periodic bed form on the surface waves generated due to a rolling oscillation of a vertical barrier either partially immersed or completely submerged in water of non uniform finite depth is investigated. A simplified perturbation technique involving a non dimensional parameter characterizing the smallness of the bottom deformation is applied to reduce the given boundary value problem to two independent boundary value problems upto first order. The first boundary value problem corresponds to the problem of water wave generation due to rolling oscillation of a vertical barrier either partially immersed or completely submerged in water of uniform finite depth. This is a well known problem whose solution is available in the literature. From the second boundary value problem, the first order correction to the wave amplitude at infinity is evaluated in terms of the shape function characterizing the bottom undulation, by employing Green's integral theorem. For a patch of sinusoidal ripples at the sea bottom, the first order correction to the wave amplitude at infinity for both the configuration of the barrier is then evaluated numerically and illustrated graphically for various values of the wave number. It is observed that resonant interaction of the wave generated, with the sinusoidal bottom undulation occurs when the ratio of twice the wavelength of the sinusoidal ripple to the wave length of waves generated, approaches unity. Also it is found that the resonance increases as the length of the barrier increases.

**Keywords:** bottom undulation; rolling oscillation; partially immersed barrier; submerged plate

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## 1 Introduction

The interaction of linear waves with a thin floating plate present in laterally unbounded sea can be used the depth of the sea is not constant. Due to this reason the effect of irregular bottom topography on waves are studied. The interaction of incident waves with irregular bottom topography of sea bed finds its importance in understanding the mechanism of wave induced mass transport that forms sand ripples of some wavelength. If this wavelength is half that of the incident wave train then these ripples produce a resonant behaviour in the reflected waves. A significant research both theoretical and experimental, has been carried out in this direction (Davies, 1982; Davies and Heathershaw, 1984; Fitz-Gerald, 1976; Roseau, 1976). In all above studies, the bottom undulation was the only obstacle in the propagation of waves. Later on, the study of the behaviour of surface waves due to irregular bottom topography was modified by including the effect of a thin floating plate. These problems were studied when the floating plate is in form of a very large floating structures (Wang and Meylan, 2002; Belibassakis and Athanassoulis, 2005; Watanabe *et al.*, 2004).

In recent past, Mandal and Gayen (2006) and Mandal and De (2007) studied the effect of a thin floating plate in form of a breakwater either partially immersed or submerged in sea with variable bottom topography, on the surface waves incident on it. They used a perturbation technique together with Green's integral theorem and Galerkin technique to obtain the reflection and transmission coefficients.

In the present paper we have investigated the effect of irregular bottom topography on the wave generated by rolling oscillation of a vertical barrier either partially immersed or completely submerged in water. Problems concerning generation of waves due to rolling oscillation of a vertical barrier partially immersed in deep water was studied long back in 1948 by Ursell. He used singular integral equation formulation to obtain closed form solution of this problem. Later Evans (1970) and Banerjee and Mandal (1992) studied the problem of generation of waves due to rolling of a submerged vertical barrier in deep water. Evans (1970) used complex variable method while Banerjee and Mandal (1992) used integral equation formulation to obtain the closed form solution. These problems are among a limited number of problems which admit of closed form solution. Later Banerjee *et al.* (1996) and (1997) considered the effect of finite depth of water region on the waves due to rolling of a vertical barrier partially immersed or completely submerged in water of uniform finite depth. They used two different methods to obtain the amplitude of the waves

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produced. The first method involves the eigen function expansion of the velocity potential describing the ensuing motion in water while the other method involves a hypersingular integral equation formulation. The two methods produce almost the same result for the amplitude at infinity of the radiated waves for both the configurations of the barrier.

In the present paper we have applied a perturbation technique (Mandal and Gayen, 2006; Mandal and De, 2007) in terms of a parameter describing smallness of the bottom undulation to reduce the boundary value problem to two independent boundary value problems upto first order. The first boundary value problem corresponds to the known problem of wave generation by a vertical barrier either partially immersed (Banerjee *et al.*, 1996) or completely submerged (Banerjee *et al.*, 1997) in water of uniform depth  $h$ . From the second boundary value problem the first order correction to the wave amplitude at infinity for both the configurations of the barrier is obtained by applying suitably the Green's integral theorem. The numerical results for the wave amplitude at infinity is illustrated for different values of wave number when the sea bottom has a patch of sinusoidal ripples. It is observed that when the wavelength of the waves generated is equal to twice the wavelength of the sinusoidal ripples at the sea bed, a resonant exact is seen to occur. Also, this resonant exact is more pronounced as the length of the barrier increases.

## 2 Formulation of the boundary value problem

We consider two dimensional motion due to rolling oscillation of a thin rigid plate described by  $x=0, y \in L$ , present in water region with a small bottom undulation described by  $y=h+\varepsilon c(x)$ . Here  $h$  is the mean depth of water,  $\varepsilon$  is a small non dimensional parameter,  $c(x)$  is a function of compact support describing the shape of the bottom.

Here we choose rectangular cartesian coordinate system in which  $y$  axis is directed vertically downwards and  $x$  axis is along the mean free surface. The plate is hinged at a point  $(0, s)$ ,  $s \in L$  and is forced to perform simple harmonic oscillation about its mean vertical position with amplitude  $\theta = \text{Re}\{\theta_0 e^{-i\sigma t}\}$ ,  $\sigma$  being the frequency of oscillation and  $\theta_0$  is small. We have considered the problem for two configurations of the barrier, namely, (i) partially immersed barrier for which  $L=[0, a]$  and (ii) completely submerged plate for which  $L=[a, b]$ . Assuming the motion to be irrotational, it can be described by the velocity potential  $\text{Re}\{\phi(x, y)e^{-i\sigma t}\}$ , where  $\phi$  satisfies the following boundary value problem.

$$\nabla^2 \phi = 0 \text{ in the water region} \quad (1)$$

$$K\phi + \phi_y = 0 \quad y=0, \quad K = \frac{\sigma^2}{g} \quad (2)$$

$g$  being the acceleration due to gravity,

$$\phi_x = i\sigma\theta_0(y-s) \quad x=0, y \in L, s \in L, s < y \quad (3)$$

$$\phi_n = 0 \quad y = h + \varepsilon c(x) \quad (4)$$

$n$  denotes the outward drawn normal,  $r^{\frac{1}{2}}\nabla\phi$  is bounded as

$$r = \{(x-c_1)^2 + (y-d_1)^2\}^{\frac{1}{2}} \rightarrow 0 \quad (5)$$

Here  $(c_1, d_1) \equiv (0, a)$  or  $(a, b)$  according as the barrier is partially immersed or submerged.

The far field conditions satisfied by  $\phi$  are

$$\phi(x, y) = \begin{cases} A\Phi_0(x, y), x \rightarrow \infty \\ B\Phi_0(-x, y), x \rightarrow -\infty \end{cases} \quad (6)$$

where  $A$  and  $B$  are unknowns which represent the amplitudes of radiated waves at infinity, and

$$\Phi_0(x, y) = \zeta(y)e^{ik_0 x} \quad (7)$$

$$\zeta(y) = \frac{\cosh k_0(h-y)}{\cosh k_0 h} \quad (8)$$

$k_0$  is the unique real positive root of the equation

$$k \tanh kh = K \quad (9)$$

## 3 Method of solution

The bottom condition (4) can be expressed approximately as

$$\phi_y - \varepsilon \frac{d}{dx} \{c(x)\phi_x\} + O(\varepsilon^2) = 0 \quad \text{on } y=h \quad (10)$$

The condition (10) implies that we can adopt the following expansion of  $\phi(x, y, \varepsilon)$ ,  $A(\varepsilon)$ ,  $B(\varepsilon)$  as

$$\begin{cases} \phi(x, y, \varepsilon) = \phi_0(x, y) + \varepsilon\phi_1(x, y) + O(\varepsilon^2) \\ A(\varepsilon) = A_0 + \varepsilon A_1 + O(\varepsilon^2) \\ B(\varepsilon) = B_0 + \varepsilon B_1 + O(\varepsilon^2) \end{cases} \quad (11)$$

Substituting (11) in the boundary value problem (1)–(3), (5), (6) and (10), we find after equating the coefficients of  $\varepsilon^0$  and  $\varepsilon$  from both sides, that  $\phi_0$  and  $\phi_1$  satisfy the following boundary value problems BVPI and BVPII.

### BVPI

The function  $\phi_0(x, y)$  satisfies

$$\nabla^2 \phi_0 = 0 \quad \text{in} \quad 0 \leq y \leq h$$

$$K\phi_0 + \phi_{0y} = 0 \quad \text{on} \quad y=0$$

$$\phi_{0x} = i\sigma\theta_0(y-s) \quad \text{on} \quad x=0, y \in L$$

$$\phi_{0y} = 0 \quad \text{on} \quad y=h$$

$$r^{\frac{1}{2}}\nabla\phi_0 \text{ is bounded as } r \rightarrow 0$$

The far field conditions for  $\phi_0$  are

$$\phi_0(x, y) \sim \begin{cases} A_0 \Phi_0(x, y), x \rightarrow \infty \\ B_0 \Phi_0(-x, y), x \rightarrow -\infty \end{cases} \quad (12)$$

**BVPII**

$\phi(x, y)$  satisfies

$$\begin{aligned} \nabla^2 \phi &= 0 \quad \text{in} \quad 0 \leq y \leq h \\ K\phi + \phi_y &= 0, \text{ on } y=0 \\ \phi_x &= 0 \text{ on } x=0, y \in L \\ \phi_y &= \frac{d}{dx}[c(x)\phi_0(x)] \text{ on } y=h \\ r^{\frac{1}{2}} \nabla \phi &\text{ is bounded as } r \rightarrow 0 \end{aligned}$$

The far field conditions are

$$\phi \sim \begin{cases} A_1 \Phi_0(x, y), x \rightarrow \infty \\ B_1 \Phi_0(-x, y), x \rightarrow -\infty \end{cases} \quad (13)$$

It may be noted here that BVPI corresponds to the known problem of generation of waves due to rolling oscillation of a vertical plate present in water of uniform finite depth  $h$ .

For the problem involving a partially immersed barrier,  $L=[0, a]$  and the solution of BVPI is given in Banerjee, Dolai and Mandal (1996). For the problem involving a completely submerged barrier,  $L=[a, b]$  and the solution is given in Banerjee, Dolai and Mandal (1997). For completeness we derive the expression for  $B_0$  for the two problems in Appendix.

Now we will proceed to obtain  $A_1, B_1$  from BVPII for (i)  $L=[0, a]$ , (ii)  $L=[a, b]$ .

Let us denote  $\text{Re}\{\psi(x, y)e^{-i\sigma t}\}$  to be the velocity potential corresponding to the problem of scattering of an incoming wave by a vertical barrier either partially immersed or completely submerged in water of finite uniform depth  $h$ , where  $\psi(x, y)$  satisfies the following boundary value problem.

$$\begin{aligned} \nabla^2 \psi &= 0 \quad \text{in} \quad 0 \leq y \leq h \\ K\psi + \psi_y &= 0 \text{ on } y=0 \\ \psi_x &= 0 \text{ on } x=0, y \in L \\ r^{\frac{1}{2}} \nabla \psi &\text{ is bounded as } r \rightarrow 0 \\ \frac{\partial \psi}{\partial y} &= 0 \quad y=h \end{aligned}$$

$$\psi(x, y) = \begin{cases} (e^{ik_0 x} + R e^{-ik_0 x}) \psi_0(y), x \rightarrow -\infty \\ T e^{ik_0 x} \psi_0(y), x \rightarrow \infty \end{cases} \quad (14)$$

where

$$\psi_0(y) = N_0^{-\frac{1}{2}} \cosh k_0(h-y) \quad (15)$$

with

$$N_0 = \frac{2k_0 h + \sinh 2k_0 h}{4k_0 h} \quad (16)$$

and  $k_0$  being unique real positive root of Eq.(9).

Also  $\psi(x, y)$  can be expressed as (Mandal and Gayen, 2006; Mandal and De, 2007)

$$\psi(x, y) = \begin{cases} (e^{ik_0 x} + R e^{-ik_0 x}) \psi_0(y) + \sum_{n=1}^{\infty} C_n e^{k_n x} \psi_n(y), x < 0 \\ T e^{ik_0 x} \psi_0(y) + \sum_{n=1}^{\infty} D_n e^{-k_n x} \psi_n(y), x > 0 \end{cases} \quad (17)$$

where  $\pm i k_n (n=1, 2, \dots)$  are the purely imaginary roots of Eq.(9),  $C_n, D_n (n=1, 2, \dots)$  are unknown constants and

$$\psi_n(y) = N_n^{-\frac{1}{2}} \cos k_n(h-y) \quad (18)$$

with

$$N_n = \frac{2k_n h + \sin 2k_n h}{4k_n h}$$

it can be shown that

$$\begin{cases} R = 1 - T \\ C_n = -D_n \end{cases} \quad (19)$$

where  $R$  and  $T$  are the unknown reflection and transmission coefficients.

Now we proceed to evaluate  $A_1, B_1$  when the barrier is partially immersed and completely submerged.

**Case I: Partially Immersed Barrier,  $L=[0, a]$** 

To obtain  $A_1$  and  $B_1$ , in this case we apply Green's integral theorem to the functions  $\phi(x, y)$  and  $\psi(x, y)$  in the region bounded by the lines

$$\begin{aligned} y=0, 0 < x \leq X; x=X, 0 \leq y \leq h; y=h, -X \leq x \leq X; \\ x=-X, 0 \leq y \leq h; y=0, -X \leq x < 0; x=0 \pm, 0 \leq y \leq a. \end{aligned}$$

We obtain after making  $X$  to tend to infinity,

$$2ik_0 h B_1 = N_0^{-\frac{1}{2}} \cosh k_0 h \int_{-\infty}^{\infty} c(x) \psi_x(x, h) \phi_{0x}(x, h) dx \quad (20)$$

Similar application of Green's integral theorem to  $\chi(x, y) = \psi(-x, y)$  and  $\phi(x, y)$  in the same region and making  $X \rightarrow \infty$ , we get

$$2ik_0 h A_1 = -N_0^{-\frac{1}{2}} \cosh k_0 h \int_{-\infty}^{\infty} c(x) \psi_x(-x, h) \phi_{0x}(x, h) dx \quad (21)$$

**Case II: Completely Submerged Plate,  $L=[a, b]$** 

In this case, to obtain  $A_1$  and  $B_1$ , we apply Green's integral theorem to the functions  $\phi(x, y)$  and  $\psi(x, y)$  in the following region by the lines

$$y = 0, -X \leq x \leq X; x = \pm X, 0 \leq y \leq h; -X \leq x \leq X, \\ y = h; x = 0 \pm a, a \leq y \leq b.$$

Making  $X \rightarrow \infty$ , we obtain

$$2ik_0 h B_1 = N_0^{-\frac{1}{2}} \cosh k_0 h \int_{-\infty}^{\infty} c(x) \psi_x(x, h) \phi_x(x, h) dx \quad (22)$$

Similar application of Green's integral theorem in the region described above  $\chi(x, y) = \psi(-x, y)$  and  $\phi(x, y)$  and making  $X \rightarrow \infty$ , we obtain

$$2ik_0 h A_1 = -N_0^{-\frac{1}{2}} \cosh k_0 h \int_{-\infty}^{\infty} c(x) \psi_x(-x, h) \phi_x(x, h) dx \quad (23)$$

In the expression for  $A_1$  and  $B_1$  given by Eq.(20)–(23), we need to know  $\psi(x, y)$ ,  $\phi(x, y)$  and  $c(x)$ . A suitable representation for  $\psi(x, y)$  is given by (17).

Also we use the representation of  $\phi(x, y)$  as (cf Banerjee, Dolai and Mandal, 1996, 1997)

$$\phi(x, y) = \begin{cases} A_0 \Phi_0(x, y) + \sum_{n=1}^{\infty} P_n \cos k_n (h - y) e^{-k_n x}, & x > 0 \\ B_0 \Phi_0(-x, y) + \sum_{n=1}^{\infty} Q_n \cos k_n (h - y) e^{k_n x}, & x < 0 \end{cases} \quad (24)$$

where

$$\begin{cases} A_0 = -B_0 \\ P_n = -Q_n \end{cases} \quad (25)$$

For completeness, we have obtained  $A_0$ ,  $B_0$ ,  $P_n$ ,  $Q_n$  in Appendix and  $\Phi_0(x, y)$  is given by Eq.(7). For describing bottom undulation at sea bed we take the shape function  $c(x)$  as

$$c(x) = \begin{cases} c_0 \sin \lambda x, & -\frac{m\pi}{\lambda} \leq x \leq \frac{m\pi}{\lambda} \equiv -l \leq x \leq l \\ 0, & \text{otherwise} \end{cases} \quad (26)$$

where  $m$  is a positive integer. This represents a patch of  $m$  sinusoidal ripples with wave number  $\lambda$ .

Substituting  $\phi$  and  $\psi$  from Eq.(17) and Eq.(24) and using Eq.(19) and Eq.(25), we obtain from Eq.(20) and Eq.(23).

$$\begin{aligned} \frac{2ik_0 h B_1 N_0^{-\frac{1}{2}}}{\cosh k_0 h} &= -\frac{2ik_0 h A_1 N_0^{-\frac{1}{2}}}{\cosh k_0 h} = \\ &= \int_{-l}^l c(x) \{ik_0 (e^{ik_0 x} - R e^{-ik_0 x}) \psi_0(h) + \\ &+ \sum_{m=1}^{\infty} C_m k_m e^{k_m x} \psi_m(h)\} \left\{ -\frac{ik_0 B_0}{\cosh k_0 h} e^{-ik_0 x} + \right. \\ &+ \sum_{n=1}^{\infty} Q_n k_n e^{k_n x} \} dx + \int_0^l c(x) \{ (1-R) ik_0 e^{ik_0 x} \psi_0(h) + \sum_{m=1}^{\infty} C_m k_m e^{-k_m x} \psi_m(h) \} \\ &\left\{ -\frac{ik_0 B_0}{\cosh k_0 h} e^{ik_0 x} + \sum_{n=1}^{\infty} Q_n k_n e^{-k_n x} \right\} dx \end{aligned}$$

Using the expression for  $c(x)$  given by Eq.(26) and noting

that  $c(x)$  is an odd function of  $x$  we obtain after simplification

$$\frac{2ik_0 h B_1 N_0^{-\frac{1}{2}}}{\cosh k_0 h} = -\frac{2ik_0 h A_1 N_0^{-\frac{1}{2}}}{\cosh k_0 h} = \quad (27)$$

$$c_0 \int_0^l \sin \lambda x \left[ \frac{k_0^2 B_0 \psi_0(h)}{\cosh k_0 h} (e^{2ik_0 x} - 1) - 2k_0 \psi_0(h) \sin k_0 x \sum_{n=1}^{\infty} Q_n k_n e^{-k_n x} \right] dx$$

Now on integration we obtain  $B_1$  as

$$\begin{aligned} 2ik_0 h B_1 N_0^{-\frac{1}{2}} &= \frac{c_0 B_0 k_0^2}{2N_0^{-\frac{1}{2}}} \left[ -\frac{\cos(\lambda + 2k_0)l}{\lambda + 2k_0} - \frac{\cos(\lambda - 2k_0)l}{\lambda - 2k_0} + \right. \\ &+ i \frac{\sin(\lambda - 2k_0)l}{\lambda - 2k_0} - i \frac{\sin(\lambda + 2k_0)l}{\lambda + 2k_0} + \frac{2\lambda}{\lambda^2 - 4k_0^2} \left. \right] + \frac{c_0 B_0 k_0^2}{N_0^{-\frac{1}{2}}} \left[ \frac{\cos \lambda l - 1}{\lambda} \right] - \\ &= \frac{c_0 k_0 \cosh k_0 h}{N_0^{-\frac{1}{2}}} \sum_{n=1}^{\infty} Q_n k_n \left[ \frac{(\lambda - k_0) \sin(\lambda - k_0)l - k_n \cos(\lambda - k_0)l}{(\lambda - k_0)^2 + k_n^2} - \right. \\ &\left. \frac{(\lambda + k_0) \sin(\lambda + k_0)l - k_n \cos(\lambda + k_0)l}{(\lambda + k_0)^2 + k_n^2} \right] e^{-k_n l} + \\ &\left. \frac{k_n}{(\lambda - k_0)^2 + k_n^2} - \frac{k_n}{(\lambda + k_0)^2 + k_n^2} \right] \end{aligned} \quad (28)$$

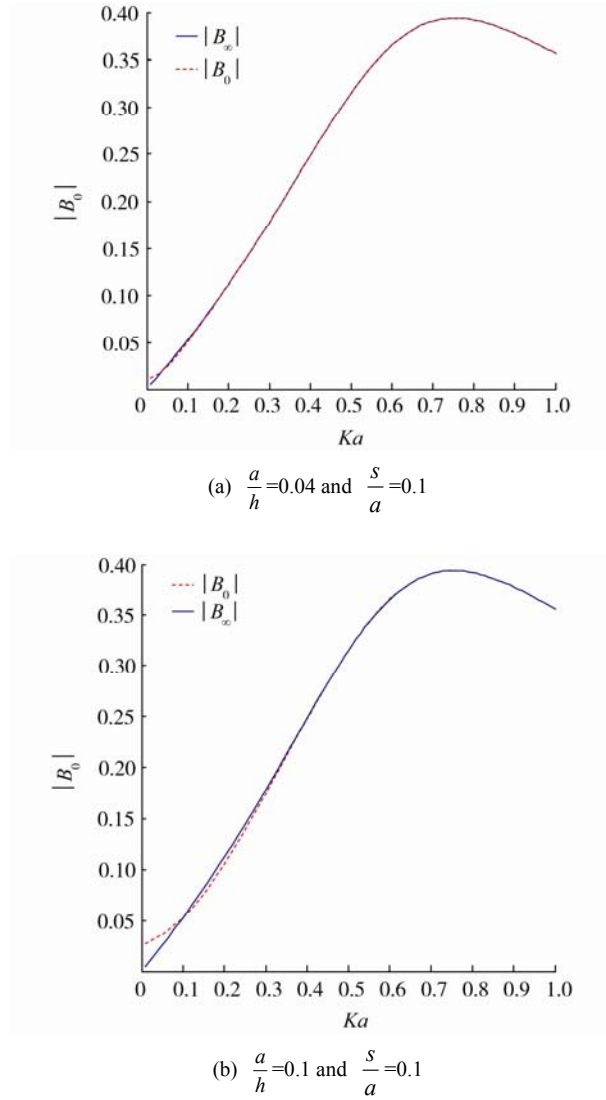
For partially immersed barrier i.e. for  $L = [0, a]$ , the expression for  $B_0$  and  $Q_n$  are substituted from (A20) to (A22) into (28) to get  $B_1$ . Similarly for completely submerged barrier i.e. for  $[a, b]$  the expression for  $B_0$  and  $Q_n$  are substituted from (A33) to (A35) into (28) to get  $B_1$ . In the next section we evaluate  $|B_1|$  numerically and discuss the numerical results.

## 4 Numerical result

For numerical evaluation of first order correction to wave amplitude  $|B_1|$  we need to know the value of  $|B_0|$  and the constants  $|Q_n|$  in the expression for  $\phi(x, y)$  given by Eq.(24). A multiterm Galerkin approximation is used to evaluate these quantities when the barrier is (i) partially immersed and (ii) submerged in water of uniform finite depth. In our numerical computation we have chosen the value of  $\lambda h$  to be unity and  $\frac{c_0}{h} = 0.1$ .

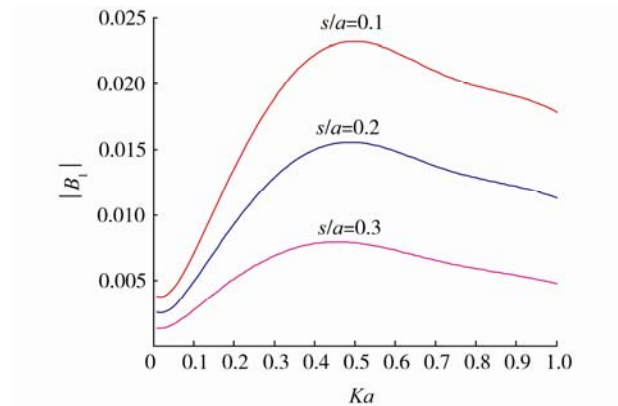
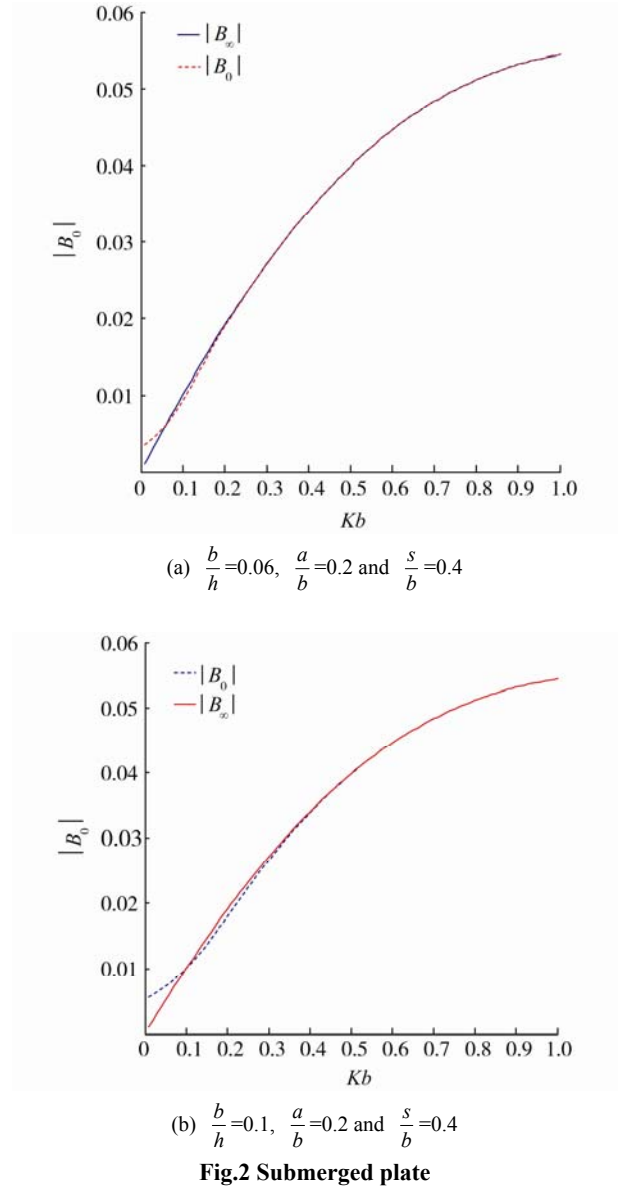
In Fig.1, the graph of  $|B_0|$ , the wave amplitude at infinity due to rolling oscillation of a partially immersed plate in water of finite depth  $h$  (in nondimensional form) is drawn for various values of the wave number  $Ka$ , for  $\frac{a}{h} = 0.04, 0.1$  and  $\frac{s}{a} = 0.1$ . Here the results for  $|B_0|$  is compared with the value of  $|B_{\infty}|$ , where  $B_{\infty}$  is the wave amplitude due to

rolling oscillation of a barrier partially immersed in deep water (Ursell, 1948). It is observed that the curve of  $|B_0|$  matches almost exactly with that of the curve for  $|B_\infty|$  when  $\frac{a}{h}=0.04, 0.1$ . Thus it is observed that the far field amplitude when the depth of water is constant for both the configurations of the barrier is consistent with the results when the depth of water is large.

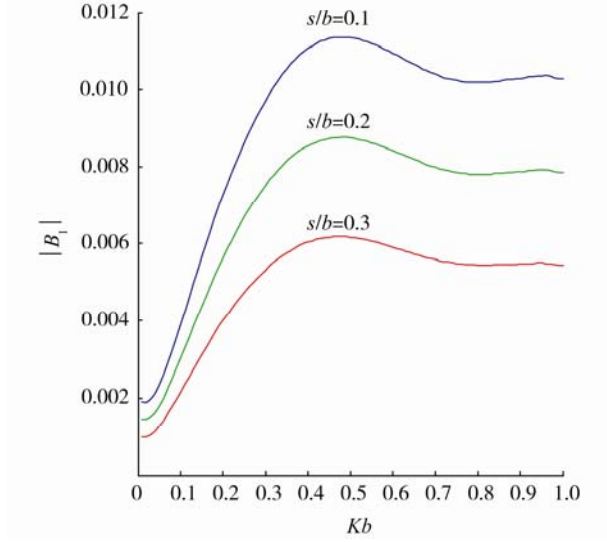


**Fig.1 Partially immersed barrier**

A similar comparison between  $|B_0|$  and  $|B_\infty|$  (Evans, 1970) is illustrated in Fig.2 when the barrier is submerged in water of uniform finite and infinite depth respectively for the values of the parameters  $\frac{a}{b}=0.2$ ,  $\frac{s}{a}=0.4$  and  $\frac{b}{h}=0.06$  and 0.1. As expected, a very good matching of these two results is observed from Fig.2 for these values of the parameters.

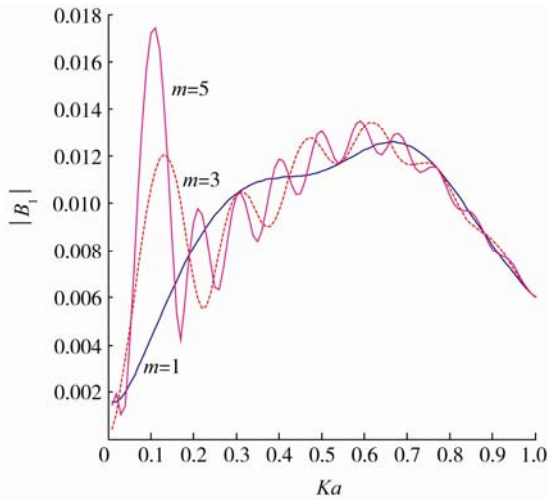


**Fig.3  $\frac{b}{h}=0.6, m=1, \frac{c}{h}=0.1$  for partially immersed barrier**

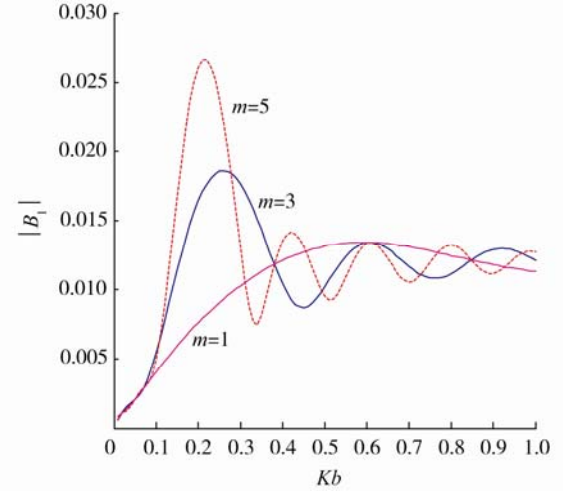


**Fig.4**  $\frac{b}{h}=0.6$ ,  $\frac{a}{b}=0.1$ ,  $m=1$  and  $\frac{c}{h}=0.1$  for submerged plate

Fig.3 depicts  $|B_1|$  against  $Ka$  for different values of  $\frac{s}{a}$  when  $\frac{a}{h}=0.6$ ,  $m=1$  in the case when barrier is partially immersed. Also in Fig.4,  $|B_1|$  is drawn against  $Kb$  for different values of  $\frac{s}{b}$ , and  $\frac{a}{h}=0.6$ ,  $\frac{a}{b}=0.1$ ,  $m=1$  when the barrier is submerged. From both the figures it is observed that  $|B_1|$  increases first, and then decreases for large values of the wave number for any value of  $\frac{s}{a}$  or  $\frac{s}{b}$ . Also it is observed in both cases that for a single ripple and for fixed length of the barrier, the lowering of the position of hinge point of the barrier causes decrease in the amplitude  $|B_1|$ .

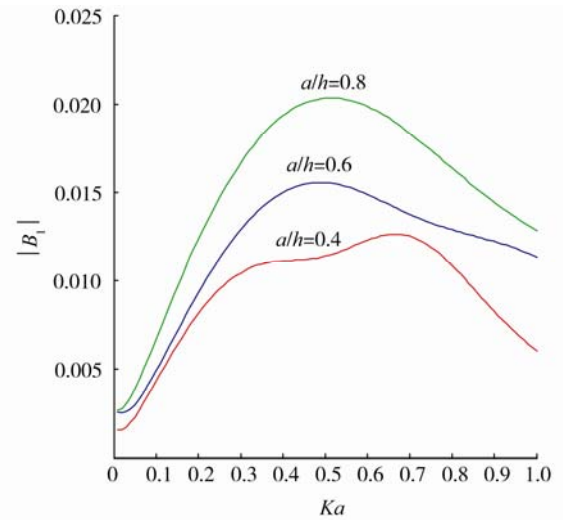


**Fig.5**  $\frac{a}{h}=0.4$ ,  $\frac{s}{a}=0.2$  and  $\frac{c}{h}=0.1$  for partially immersed barrier

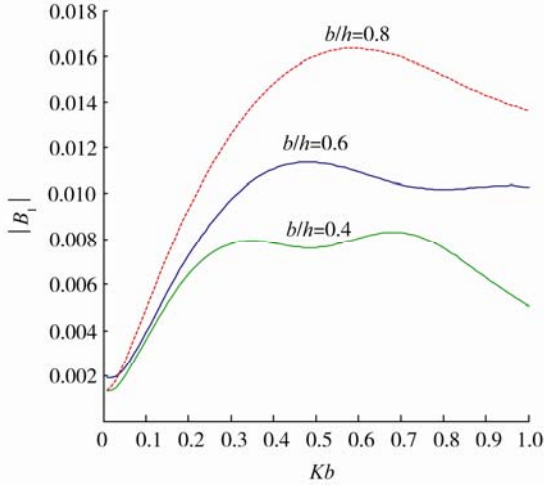


**Fig.6**  $\frac{b}{h}=0.8$ ,  $\frac{a}{b}=0.2$ ,  $\frac{s}{b}=0.1$  and  $\frac{c}{h}=0.1$  for submerged plate

Fig.5 and Fig.6 depict  $|B_1|$  against various values of the wave number for different values of  $m$ . In Fig.5,  $|B_1|$  is plotted against  $Ka$  for  $\frac{a}{h}=0.4$ ,  $\frac{s}{a}=0.2$  and for  $m=1,3,5$  when the barrier is partially immersed. In fig6,  $|B_1|$  is drawn against  $Kb$  for  $\frac{b}{h}=0.8$ ,  $\frac{a}{b}=0.2$ ,  $\frac{s}{b}=0.1$  and for  $m=1, 3, 5$  when the barrier is submerged. In both the figures, the oscillatory nature of  $|B_1|$  is observed as the number of ripples increases. This phenomena may be attributed due to multiple interaction of the waves generated with the sand ripples, barrier and the free surface.

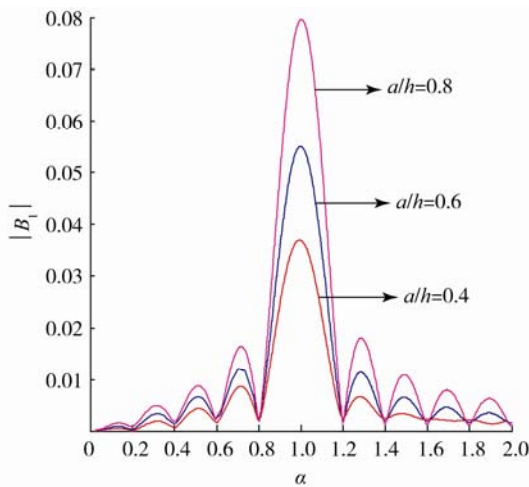


**Fig.7**  $\frac{s}{a}=0.2$  and  $m=1$  for partially immersed barrier

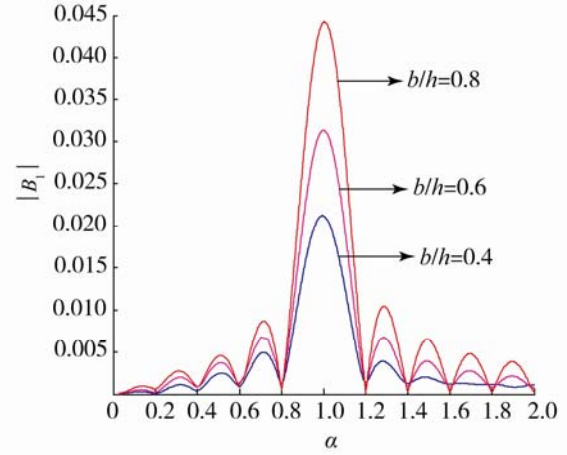


**Fig.8**  $\frac{a}{b}=0.1$ ,  $\frac{s}{b}=0.1$  and  $m=1$  for submerged plate

Fig.7 and Fig.8 show the effect of the length of the barrier on  $B_1$  for different values of wave number for a single sand ripple and for a fixed hinge point for a partially immersed barrier. In Fig.7,  $|B_1|$  is plotted against  $Ka$  for  $\frac{s}{a}=0.2$ ,  $m=1$  and for  $\frac{a}{h}=0.4, 0.6, 0.8$  when the barrier is partially immersed. In Fig.8,  $|B_1|$  is depicted against  $Kb$  for  $\frac{a}{b}=0.1$ ,  $\frac{s}{b}=0.1$ ,  $m=1$  and  $\frac{b}{h}=0.4, 0.6, 0.8$  when the barrier is submerged. From both figures it is observed that  $|B_1|$  increases at first, then shows a mild oscillation and then decreases as the wave number increases. Also, the increase in the length of the barrier causes increase in amplitude  $|B_1|$ .



**Fig.9**  $Ka=0.3$ ,  $\frac{s}{a}=0.2$  and  $m=10$  for partially immersed barrier



**Fig.10**  $\frac{a}{b}=0.1$ ,  $Kb=0.3$ ,  $\frac{s}{b}=0.2$  and  $m=10$  for submerged plate

Fig.9 and Fig.10 depict  $|B_1|$  against  $\alpha = \frac{2k_0}{\lambda}$  for different lengths of the barrier. In fig9,  $|B_1|$  is plotted against  $\alpha$  for  $\frac{a}{h}=0.4, 0.6, 0.8$ ,  $Ka=0.3$ ,  $\frac{s}{a}=0.2$ ,  $m=10$  in the case when the barrier is partially immersed. Also when the barrier is submerged,  $|B_1|$  is illustrated against  $\alpha$  for  $\frac{b}{h}=0.4, 0.6, 0.8$ ,  $\frac{a}{b}=0.1$ ,  $\frac{s}{b}=0.2$  and  $m=10$ . In both cases it is observed that  $|B_1|$  shows oscillatory behavior and attains a peak for  $\alpha \approx 1$  when the number of sinusoidal ripple in sea bed is 10. This phenomena is featured as Bragg resonance and is due to resonant interaction between the wave generated and the ripples in the sea bed. This resonance occurs when the wave length of the sinusoidal ripple in the sea bed becomes half the wave length of the waves generated due to rolling oscillation of the barrier. It is also seen from both the figures that the resonance increases as the length of barrier increases. This shows that the length of the barrier has some impact upon the resonant interaction between sea bed undulation and the wave generated due to rolling oscillation.

## 5 Conclusions

In the present paper the effect of a small bottom undulation on the waves generated due to rolling of a vertical plate either partially immersed or submerged in water is studied here. A simplified perturbation analysis is applied to reduce the given boundary value problem to two independent boundary value problems upto first order. The solution of one of the boundary value problem is well known in the literature and it corresponds to the problem of generation of waves due to rolling oscillation of a vertical barrier either partially immersed or submerged in water of uniform finite depth  $h$ . The first order correction to amplitude of waves at

infinity is obtained by an application of Green's integral theorem in terms of the function characterising the bottom undulation. It is observed that a patch of sinusoidal ripples at the bottom produces a resonant effect in waves when wavelength of waves generated is equal to twice the wavelength of sinusoidal ripples at sea bed. The resonant effect increases with the length of the barrier.

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## Appendix

For completeness, we derive here the solution of BVPI for both the cases when the barrier is partially immersed for which  $L=[0,a]$  and submerged for which  $L=[a,b]$  in water of uniform finite depth.

We consider the representation of  $\phi_0(x, y)$  as

$$\phi_0(x, y) = \begin{cases} A_0 \Phi_0(x, y) + \sum_{n=1}^{\infty} P_n \cos k_n(h-y) e^{-k_n x}, & x > 0 \\ B_0 \Phi_0(-x, y) + \sum_{n=1}^{\infty} Q_n \cos k_n(h-y) e^{k_n x}, & x < 0 \end{cases} \quad (A1)$$

where  $k_n$  ( $n=1, 2, \dots$ ) are the purely imaginary roots of

$$k \tan kh + K = 0$$

and  $A_0$ ,  $B_0$ ,  $P_n$  and  $Q_n$  are unknowns.

Let  $G(x, y; \xi, \eta)$  be the velocity potential due to a line source situated at  $(\xi, \eta)$  ( $0 < \eta < h$ ). Then  $G(x, y; \xi, \eta)$  is given as (Thorne, 1953)

$$G(x, y; \xi, \eta) = -\ln \frac{\rho}{\rho^|} + 2 \int_{c_1}^{\infty} \frac{\exp\{-k(y+\eta)\}}{k-K} \cos k(x-\xi) dk + 2 \int_{c_2}^{\infty} \frac{L(k, y) L(k, \eta) \exp(-kh)}{k(k-K) \Delta(k)} \cos k(x-\xi) dk \quad (A2)$$

where

$$\rho, \rho^| = \{(x-\xi)^2 + (y \mp \eta)^2\}^{\frac{1}{2}} \quad (A3)$$

$$L(k, y) = k \cosh ky - K \sinh ky \quad (A4)$$

$$\Delta(k) = k \sinh kh - K \cosh kh \quad (A5)$$

**Case I:** partially immersed barrier,  $L \equiv [0, a]$

Applying Green's integral theorem to the functions  $\phi_0(x, y)$  and  $G(x, y; \xi, \eta)$  in the region

$$y = 0, 0 \leq x \leq X; y = 0, -X \leq x \leq 0; x = -X, 0 \leq y \leq Y; y = Y, -X \leq x \leq X; x = X, 0 \leq y \leq Y; x = 0 \pm, 0 \leq y \leq a$$

and a small circle of radius  $\varepsilon_1$  about  $(\xi, \eta)$ , we obtain as  $X, Y \rightarrow \infty$  as  $\varepsilon_1 \rightarrow 0$ .

$$\phi_0(\xi, \eta) = \frac{1}{2\pi} \int_L f(y) G_x(0, y; \xi, \eta) dy \quad (A6)$$

with

$$f(y) = \phi_0(+0, y) - \phi_0(-0, y) \quad (A7)$$

so that

$$f(a) = 0 \quad (A8)$$

Noting the condition that  $\phi_{0x} = i\sigma\theta_0(\eta-s)$  and  $\xi=0$ ,  $\eta \in L$ , we get from (A6)

$$\int_B f(y) G_{x\xi}(0, y; 0, \eta) dy = 2\pi i \sigma \theta_0(\eta-s) \quad \eta \in L \quad (A9)$$

Substituting  $G_{x\xi}(0, y; 0, \eta)$  from (A2) into (A9) and



substituting  $y = ap$  and  $\eta = aq$  in the result thus obtained, we get

$$\int_0^1 F(p) \left[ \frac{1}{(p-q)^2} + \chi_1(p, q) \right] dp = H(q) \quad 0 < q < 1 \quad (A10)$$

where  $\chi_1(p, q)$  is given by

$$\begin{aligned} \chi_1(p, q) = & \frac{1}{(p+q)^2} + \frac{2Ka}{p+q} - 2(Ka)^2 \exp(-\mu K) \\ & [\ln \mu K + \gamma + \sum_{m=1}^{\infty} \frac{(\mu K)^m}{m!m}] + \\ & 2a^2 i \int_0^{\infty} \frac{L\{r \exp(\frac{i\pi}{4}), ap\} L\{r \exp(\frac{i\pi}{4}), aq\} \exp\{-r h \exp(\frac{i\pi}{4})\}}{\{r \exp(\frac{i\pi}{4}) - K\} \Delta\{r \exp(\frac{i\pi}{4})\}} dr + \\ & 8\pi i k_0 a^2 \frac{L(k_0, ap) L(k_0, aq) \cosh k_0 h}{(k_0 - K)(2k_0 h + \sinh 2k_0 h)} \exp(-k_0 h) - \\ & 2\pi i (Ka)^2 \exp\{-Ka(p+q)\} \end{aligned} \quad (A11)$$

$$H(q) = 2\pi i \sigma \theta_0(aq - s) \quad (A12)$$

and

$$aF(p) = f(ap) \quad (A13)$$

Since  $F(1) = 0$  and  $F(0)$  are bounded, following Parsons and Martin (1994) we can assume

$$F(p) = (1-p^2)^{\frac{1}{2}} \sum_{n=0}^N a_n U_n(p) \quad (A14)$$

So that

$$F(0) = \sum_{n=0}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^n a_{2n}$$

where  $U_n(p)$  is the Tchebyshev polynomials of a second kind given by  $U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}$  and  $a_n$ 's ( $n = 0, 1, \dots, N$ ) are to be determined.

Using (A14) in (A10) we get after simplification,

$$\sum_{n=0}^N a_n c_n(q) = H(q) \quad 0 < q < 1 \quad (A15)$$

where

$$c_n(q) = \int_0^1 \frac{(1-p^2)^{\frac{1}{2}} U_n(p)}{(p-q)^2} dp + \int_0^1 (1-p^2)^{\frac{1}{2}} \chi_1(p, q) U_n(p) dp \quad (A16)$$

To find the unknown constants  $a_n$ 's we substitute  $q = q_j$  in (A15), where

$$q_j = \frac{1}{2} [\cos\{\frac{(2j+1)\pi}{2N+2}\} + 1] \quad j = 0, 1, \dots, N \quad (A17)$$

This gives

$$\sum_{n=0}^N a_n c_n(q_j) = H(q_j) \quad j = 0, 1, \dots, N \quad (A18)$$

Solving this system of linear equations by known standard method we obtain  $a_n$ 's. Knowing  $a_n$ 's, one can obtain  $F(p)$  from (A14) and hence  $f(y)$  from (A13).

To obtain the wave amplitude, we make  $\xi \rightarrow \infty$  in (A6) so that

$$A_0 = -B_0 = -\frac{2k_0 \cosh k_0 h}{2k_0 h + \sinh 2k_0 h} \int_0^a f(y) \cosh k_0(h-y) dy \quad (A19)$$

Substituting  $y = ap$  in (A19) and putting  $F(p)$ , we have

$$A_0 = -B_0 = -\frac{2k_0 a^2 \cosh k_0 h}{2k_0 h + \sinh 2k_0 h} \sum_{n=0}^N a_n d_n \quad (A20)$$

where

$$d_n = \int_0^1 (1-p^2)^{\frac{1}{2}} U_n(p) \cosh k_0(h-ap) dp \quad (A21)$$

Also from (A6) we have after substituting  $G_x(0, y; \xi, \eta)$  we obtain

$$Q_n = -P_n = -\frac{2k_n}{2k_n h + \sinh 2k_n h} \int_0^a f(y) \cos k_n(h-y) dy \quad (A22)$$

Using  $f(y)$  in (A22) one can obtain  $Q_n$  and  $P_n$ .

Finally substituting  $A_0, B_0, P_n$  and  $Q_n$  in (A1), we get  $\phi_0(x, y)$ .

**Case II:** Submerged barrier,  $L \equiv [a, b]$

We apply Green's integral theorem to  $\phi_0(x, y)$  and  $G(x, y; \xi, \eta)$  in the region  $y = 0, -X \leq x \leq X$ ;  $x = \pm X, 0 \leq y \leq Y$ ;  $y = Y, -X \leq x \leq X$ ;  $x = 0 \pm, a \leq y \leq b$  and a small circle of radius  $\varepsilon_1$  about  $(\xi, \eta)$ , we get as  $X, Y \rightarrow \infty$  and  $\varepsilon_1 \rightarrow 0$ , we get

$$\phi_0(\xi, \eta) = \frac{1}{2\pi} \int_L f(y) G_x(0, y; \xi, \eta) dy \quad (A23)$$

where  $f(y) = \phi_0(+0, y) - \phi_0(-0, y)$  and  $f(a) = f(b) = 0$ .

Noting that

$$\phi_\xi(0, \eta) = i\sigma \theta_0(\eta - s), \quad a < \eta < b$$

We get from (A23)

$$\int_L f(y) G_{x\xi}(0, y; 0, \eta) dy = 2\pi i \sigma \theta_0(\eta - s) \quad \eta \in L \quad (A24)$$

Substituting  $y = \frac{a+b}{2} + \frac{b-a}{2} p$  and  $\eta = \frac{a+b}{2} + \frac{b-a}{2} q$  in (A24), we get

$$\int_{-1}^1 F(p) \left[ \frac{1}{(p-q)^2} + \kappa(p, q) \right] dp = H(q), \quad -1 < q < 1 \quad (A25)$$

where

$$\kappa(p, q) = \frac{(b-a)^2}{4} \left[ \frac{1}{\mu^2} + \frac{2K}{\mu} - 2K^2 e^{-\mu K} \{ \ln \mu K + \gamma + \sum_{m=1}^{\infty} \frac{(\mu K)^m}{m! m} \} + 2i \int_0^{\infty} \frac{L\{re^{\frac{i\pi}{4}}, y(p)\} L\{re^{\frac{i\pi}{4}}, \eta(q)\} e^{\{-rh\frac{i\pi}{4}\}}}{\{re^{\frac{i\pi}{4}} - K\} \Delta(re^{\frac{i\pi}{4}})} dr + 8\pi i k_0 \frac{L\{k_0, y(p)\} L\{k_0, \eta(q)\} \cosh k_0 h}{(k_0 - K)(2k_0 h + \sinh 2k_0 h)} e^{-k_0 h} - 2\pi i K^2 e^{-K\mu} \right]$$

with

$$\mu = y(p) + \eta(q) = a + b + \frac{(b-a)}{2}(p+q)$$

$$H(q) = 2\pi i \sigma \theta_0 [\eta(q) - s] \quad (A27)$$

$$F(p) = \frac{2}{b-a} f[y(p)]$$

Since  $F(\pm 1) = 0$ , following Parsons and Martin (1992),  $F(p)$  can be approximated by

$$F(p) = (1-p^2)^{\frac{1}{2}} \sum_{n=0}^N a_n U_n(p) \quad (A28)$$

where  $U_n(p)$  is the Tchebyshev polynomials of a second kind and  $a_n$ 's ( $n=0, 1, \dots, N$ ) are to be determined. Using the expansion (A28) in (A25), we obtain

$$\sum_{n=0}^N a_n c_n(q) = H(q), \quad -1 < q < 1 \quad (A29)$$

where

$$c_n(q) = -\pi(n+1)U_n(p) + \int_{-1}^1 (1-p^2)^{\frac{1}{2}} \kappa(p, q) U_n(p) dp \quad (A30)$$

To find the unknown constants  $a_n$  ( $n=0, 1, \dots, N$ ), we put  $q = q_j$ ,  $j=0, 1, \dots, N$  in (A29) to obtain the linear system

$$\sum_{n=0}^N a_n c_n(q_j) = h(q_j), \quad j=0, 1, \dots, N \quad (A31)$$

where

$$q = q_j = \cos \left\{ \frac{(j+1)\pi}{N+2} \right\} \quad j=0, 1, \dots, N \quad (A32)$$

Solving this linear system one can obtain  $a_n$ 's and hence  $F(p)$  and  $f(y)$  from (A28) and (A27). Knowing  $f(y)$  we get the wave amplitude  $A_0$  and  $B_0$  as

$$A_0 = -B_0 = -\frac{1}{2} \frac{(b-a)^2 k_0 \cosh k_0 h}{(2k_0 h + \sinh 2k_0 h)} \sum_{n=0}^N a_n d_n \quad (A33)$$

where

$$d_n = \int_{-1}^1 (1-p^2)^{\frac{1}{2}} U_n(p) \cosh k_0 \left[ h - \frac{a+b}{2} - \frac{b-a}{2} q \right] dp \quad (A34)$$

Also from (A6) we get after substituting  $G_x(0, y; \xi, \eta)$  we get

$$Q_n = -P_n = -\frac{2k_n}{2k_n h + \sin 2k_n h} \int_a^b f(t) \cos k_n(h-t) dt \quad (A35)$$

Using  $f(y)$  in (A35) we get  $Q_n$  and  $P_n$ .

Substituting  $A_0, B_0, P_n$  and  $Q_n$  in (A1), we get the form of  $\phi_0(x, y)$ .



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