

Research on Uniform Array Beamforming Based on Support Vector Regression

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Abstract: An approach was proposed for optimizing beamforming that was based on Support Vector Regression (SVR). After studying the mathematical principal of the SVR algorithm and its primal cost function, the modified cost function was first applied to uniform array beamforming, and then the corresponding parameters of the beamforming were optimized. The framework of SVR uniform array beamforming was then established. Simulation results show that SVR beamforming can not only approximate the performance of conventional beamforming in the area without noise and with small data sets, but also improve the generalization ability and reduce the computation burden. Also, the side lobe level of both linear and circular arrays by the SVR algorithm is improved sharply through comparison with the conventional one. SVR beamforming is superior to the conventional method in both linear and circular arrays, under single source or double non-coherent sources.

Keywords: array beamforming; support vector regression; optimization; framework; cost function

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1 Introduction

Support vector machines (SVMs) are learning systems that use a hypothesis space of linear functions in a high dimensional feature space, trained with a learning algorithm from optimization theory that implements a learning bias derived from statistical learning theory by Vapnik (2001). SVMs are gaining popularity due to many attractive features and promising performance (Mittra *et al.*, 2004). Recently, SVM have been successfully extended to Support Vector Regression (SVR) problems (Smola and Schölkopf, 2003). The minimization principle of SVM employs the Structural Risk Minimization (SRM) principle, which has been shown to be superior to the traditional Empirical Risk Minimization (ERM) principle employed in conventional learning algorithms.

Beamforming is one of the most important branches in modern array signal processing by Van Trees (2002). Array signal processing involves complex signals processing, for which a complex-valued formulation of the SVM is needed (Ramon *et al.*, 2005; Trafalis and Gilbert, 2006). The real and imaginary parts of the error in the primal optimization are introduced to solve a complex valued constrained optimization problem. The resulting algorithm is a counterpart of the real valued SVR naturally. Therefore, researching the basic framework of the SVR algorithm as applied in array signal processing can provide a new approach for optimizing

beamforming.

2 The basic principles of support vector machines and the cost function

2.1 Basic principles of SVM

Let $\mathbf{x}[n] = [x_1(n), \dots, x_M(n)]^T \in C^{M \times 1}$ be the complex vector of M elements of array observations, which is spatially sampled data. $\mathbf{w} = [w_1, \dots, w_M]^T \in C^{M \times 1}$ is the complex vector of weights, and M is the number of array sensors. The output of a linear beamformer can be written as

$$d[n] = \mathbf{w}^T \mathbf{x}[n] + e[n] \quad (1)$$

where $d[n]$ is the output of desired beamformer and $e[n]$ is the output error. Coefficients \mathbf{w} are usually estimated through minimization of a certain cost function on $e[n]$.

The SVM approach can be applied to the adjustment of this model. The main idea of SVR is to obtain the solution which has the minimum norm of \mathbf{w} . Due to the minimization of the weight vector norm, the solution will be regularized in the sense of Thikonov, improving the generalization performance (Song *et al.*, 2009; Gaudes *et al.*, 2007).

After introducing a positive parameter \mathcal{E} , we can get the following constraints:

$$\begin{aligned} d[n] - \mathbf{w}^T \mathbf{x}[n] &\leq \mathcal{E} + \xi_n \\ -d[n] + \mathbf{w}^T \mathbf{x}[n] &\leq \mathcal{E} + \xi'_n \\ \xi_n, \xi'_n &\geq 0 \end{aligned} \quad (2)$$

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ξ_n and ξ'_n are the “slack” variables or sometimes are called losses. The optimization problem is then intended to minimize a cost function over these variables. The parameter \mathcal{E} is used as an error threshold to allow those ξ_n or ξ'_n for which the error is less than \mathcal{E} to be zero. This is equivalent to the minimization of the so-called \mathcal{E} -insensitive or Vapnik loss function as follows:

$$L_{\mathcal{E}}(\varepsilon) = \begin{cases} |e| - \mathcal{E}, & |e| > \mathcal{E} \\ 0, & |e| \leq \mathcal{E} \end{cases} \quad (3)$$

Then the function to be minimized becomes

$$L_p = \|w\|^2 + C \sum_n L_{\mathcal{E}}(\xi_n, \xi'_n) \quad (4)$$

where $\xi_n, \xi'_n \geq 0$, C is the tradeoff between the minimization of the norm which is used to improve the generalization ability and minimize the errors.

Through Lagrange multipliers α_i and α'_i the optimization of the above constrained problem leads to the dual formulation (3) to be minimized with respect to $(\alpha_i - \alpha'_i)$.

$$L_d = -(\alpha - \alpha')^T R(\alpha - \alpha') + (\alpha - \alpha')^T y[n] - (\alpha + \alpha')^T \mathbf{I} \mathcal{E} \quad (5)$$

Eq. (5) involves the Gram matrix R of the dot products of the data vector $\mathbf{x}[n]$. This matrix may be singular and thus produce an ill-conditioned problem. To avoid this numerical inconvenience, a small diagonal $\gamma \mathbf{I}$ is added to the matrix prior to the numerical optimization (Cherkassky and Ma, 2004; Mestre and Lagunas, 2006).

2.2 The cost function for SVM

Any practical regression algorithm has a loss function, which describes how the estimated function deviates from the true one. Many forms for the loss function can be found in literature: e.g. linear, quadratic, exponential loss function, etc. Here we present a modified derivation of the SVM regressor which leads to a more convenient equivalent cost function as follows:

$$L_R(e) = \begin{cases} 0 & |e| < \mathcal{E} \\ \frac{1}{2\gamma}(|e| - \mathcal{E})^2 - \mathcal{E} & \mathcal{E} \leq |e| \leq \mathcal{E} + e_c \\ C(|e| - \mathcal{E}) - \frac{1}{2}\gamma C^2 & e_c \leq |e| \end{cases} \quad (6)$$

where $e_c = \mathcal{E} + \gamma C$, γ is the diagonal loading parameter of covariance matrix R . C is the tradeoff between the empirical risk and the structural risk. $\mathcal{E} > 0$ is a predefined constant which controls the noise tolerance.

As it can be seen from Fig.1, the cost function is quadratic for the data which produce errors between \mathcal{E} and e_c , and linear for errors above e_c . The regression algorithm does not care about errors as long as they are less than \mathcal{E} , but will not accept any deviation larger than this. As mentioned above, the cost function provides a function that is numerically regularized by the matrix $\gamma \mathbf{I}$. Thus, we can adjust the parameter e_c to apply a quadratic cost for the samples which are mainly affected by thermal noise, for which the quadratic cost has maximum likelihood. The linear cost is then applied to the samples which are outliers. Using a linear cost function, the contribution of the outliers to the solution will not depend on its error value, but only on its sign, thus avoiding the bias that a quadratic cost function produces.

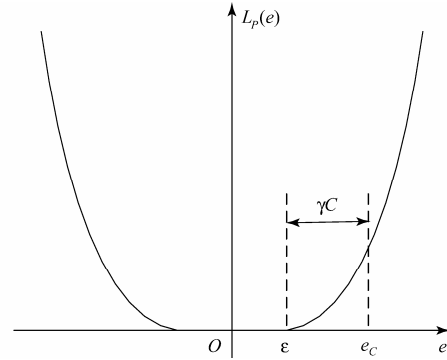


Fig.1 Cost function of SVM

3 The beamformer based on support vector machine

For an M -element array beamformer, the output vector which receives K signals can be written in matrix notation,

$$\mathbf{x}[n] = \mathbf{A}s[n] + \mathbf{g}[n] \quad (7)$$

where $\mathbf{A} = [a(\theta_1), \dots, a(\theta_K)]$, $\mathbf{a}(\theta_i) = [e^{-jk_i}, \dots, e^{-j(M-1)k_i}]^T$, $s[n] = [s_1[n], \dots, s_K[n]]^T$, $\mathbf{g}[n] = [g_1[n], \dots, g_M[n]]^T$. \mathbf{A} and $\mathbf{a}(\theta_i)$ are respectively the $M \times K$ steering matrix and vector, $s[n]$ is the received signal, and $\mathbf{g}[n]$ is the zero-mean Gaussian noise process present at the output of each array element.

The output vector $\mathbf{x}[n]$ is linearly processed to obtain the desired output $d[n]$. The expression for the output of the array processor is

$$y[n] = \mathbf{w}^T \mathbf{x}[n] = d[n] + e[n] \quad (8)$$

where $y[n]$ is actual output of beamformer, $\mathbf{w} = [w_1, \dots, w_M]$ is the weight vector of the array and $e[n]$ is the estimation error.

When nonzero empirical errors are expected and for a set of N observed samples of $\{\mathbf{x}[n]\}$, the function to be minimized is

$$\frac{1}{2} \|w\|^2 + \sum_{n=1}^N L_R(e[n], \varepsilon, \gamma, C) \quad (9)$$

where $L_R(e[n], \varepsilon, \gamma, C)$ is the modified ε -insensitive function.

Thus, according to the modified error cost function (6), the function to be minimized then becomes

$$\frac{1}{2} \|w\|^2 + \sum L_R(\xi_n + \xi'_n) + \sum L_R(\zeta_n + \zeta'_n) \quad (10)$$

subject to

$$\begin{aligned} \operatorname{Re}(d[n] - w^T x[n]) &\leq \varepsilon + \xi_n \\ \operatorname{Re}(-d[n] + w^T x[n]) &\leq \varepsilon + \xi'_n \\ \operatorname{Im}(d[n] - w^T x[n]) &\leq \varepsilon + \zeta_n \\ \operatorname{Im}(-d[n] + w^T x[n]) &\leq \varepsilon + \zeta'_n \\ \xi[n], \xi'[n], \zeta[n], \zeta'[n] &\geq 0 \end{aligned} \quad (11)$$

where ξ_n and ξ'_n stand for positive and negative errors in the real part of the output, respectively. ζ_n and ζ'_n represent the errors for the imaginary part. Also we notice that errors are either negative or positive and, therefore, only one of the losses takes a nonzero value, that is, either ξ_n or ξ'_n (either ζ_n or ζ'_n) is zero. Therefore, this constraint can be written as

$$\begin{aligned} \xi_n \xi'_n &= 0 \\ \zeta_n \zeta'_n &= 0 \end{aligned} \quad (12)$$

Finally, as in other SVR formulations, the parameter C can be seen as a tradeoff factor between the empirical risk and the structural risk.

The minimization problem in Eq.(10) is called the primal objective function. The key idea of the dual problem is to construct a Lagrange function from the primal objective function and the corresponding constraints. So it is possible to transform the minimization of the primal function (10) into the optimization of the dual function or Lagrange function subject to constraints in Eq.(11). First, we introduce the constraints into the primal function by means of Lagrange multipliers, obtaining the following primal-dual function:

$$\begin{aligned} L_{pd} = & \frac{1}{2} \|w\|^2 + C \sum_{n \in I_1} (\xi_n + \xi'_n) + C \sum_{n \in I_2} (\zeta_n + \zeta'_n) \times \\ & \frac{1}{2\gamma} \sum_{n \in I_2} (\xi_n^2 + \xi_n'^2) + \frac{1}{2\gamma} \sum_{n \in I_2} (\zeta_n^2 + \zeta_n'^2) - \\ & \sum_{n=n_0}^N (\lambda_n \xi_n + \lambda'_n \xi'_n) - \sum_{n=n_0}^N (\eta_n \zeta_n + \eta'_n \zeta'_n) \times \\ & \sum_{n=n_0}^N \alpha_n [\operatorname{Re}(d[n] - w^T x[n]) - \varepsilon - \xi_n] \times \\ & \sum_{n=n_0}^N \alpha'_n [\operatorname{Re}(-d[n] + w^T x[n]) - \varepsilon - \xi'_n] \times \\ & \sum_{n=n_0}^N \beta_n [\operatorname{Im}(d[n] - w^T x[n]) - j\varepsilon - j\zeta_n] \times \\ & \sum_{n=n_0}^N \beta'_n [\operatorname{Im}(-d[n] + w^T x[n]) - j\varepsilon - j\zeta'_n] \end{aligned} \quad (13)$$

with a dual set of variables or Lagrange multipliers constrained to $\alpha_n, \beta_n, \lambda_n, \eta_n, \alpha'_n, \beta'_n, \lambda'_n, \eta'_n \geq 0$ and with $\xi_n, \zeta_n, \xi'_n, \zeta'_n \geq 0$.

As shown in Fig.1, the cost function (6) has two active segments, a quadratic one and a linear one. The following constraints must also be fulfilled:

$$\alpha_n \alpha'_n = 0, \beta_n \beta'_n = 0 \quad (14)$$

Besides, the Karush-Kuhn-Tucker (KKT) (Gretton *et al.*, 2003) conditions force

$$\lambda_n \xi_n = 0, \lambda'_n \xi'_n = 0 \text{ and } \eta_n \zeta_n = 0, \eta'_n \zeta'_n = 0.$$

In order to get the minimization solution with respect to the dual variables, we obtain an optimal solution for the weights by minimizing L_{pd} with respect to w_i

$$w = \sum_{n=0}^N \psi_n x^*[n] \quad (15)$$

where $\psi_n = \alpha_n - \alpha'_n + j(\beta_n - \beta'_n)$, α_n, β_n are Lagrange multipliers. This result is analogous to the real-valued SVR problem, except that the Lagrange multipliers α_n and β_n for both real and imaginary components have been considered. Optimizing L_{pd} with respect to ξ_n and ζ_n and applying the KKT conditions lead to an analytical relationship between the residuals and the Lagrange multipliers. This relationship is given by

$$(\alpha - \alpha') = \begin{cases} -C, & \text{Re}(e) \leq -e_c \\ \frac{1}{\gamma}(\text{Re}(e) + \varepsilon), & -e_c \leq \text{Re}(e) \leq -\varepsilon \\ 0, & -\varepsilon \leq \text{Re}(e) \leq \varepsilon \\ \frac{1}{\gamma}(\text{Re}(e) - \varepsilon), & \varepsilon \leq \text{Re}(e) \leq e_c \\ C, & e_c \leq \text{Re}(e) \end{cases}$$

and

$$(\beta - \beta') = \begin{cases} -C, & \text{Im}(e) \leq -e_c \\ \frac{1}{\gamma}(\text{Im}(e) + \varepsilon), & -e_c \leq \text{Im}(e) \leq -\varepsilon \\ 0, & -\varepsilon \leq \text{Im}(e) \leq \varepsilon \\ \frac{1}{\gamma}(\text{Im}(e) - \varepsilon), & \varepsilon \leq \text{Im}(e) \leq e_c \\ C, & e_c \leq \text{Im}(e) \end{cases} \quad (16)$$

Using Eq.(15), the norm of the complex coefficients can be written as

$$\|w\|^2 = \sum_{i=0}^N \sum_{j=n_0}^N \psi_j \psi_i^* \mathbf{x}[j] \mathbf{x}^*[i] \quad (17)$$

By using matrix notation again and storing all partial correlations in Eq. (16), we can write

$$\mathbf{R}[j, i] = \mathbf{x}[j] \mathbf{x}^*[i] \quad (18)$$

Hence, the norm of the coefficients can be written,

$$\|w\|^2 = \boldsymbol{\psi}^H \mathbf{R} \boldsymbol{\psi} \quad (19)$$

In Eq. (18) \mathbf{R} is the matrix with elements $R[j, i]$ and $\boldsymbol{\psi} = (\psi_{n_0}, \dots, \psi_N)^T$. By substituting Eq.(14) into Eq.(12), the dual function to be maximized is as follows:

$$\begin{aligned} L_d = & \frac{1}{2} \boldsymbol{\psi}^H \mathbf{R} \boldsymbol{\psi} - \text{Re}[\boldsymbol{\psi}^H \mathbf{R}(\alpha - \alpha')] + \\ & \text{Im}[\boldsymbol{\psi}^H \mathbf{R}(\beta - \beta')] + \\ & \text{Re}[(\alpha - \alpha')^T \mathbf{y}] - \text{Im}[(\beta - \beta')^T \mathbf{y}] - \\ & (\alpha + \alpha') \mathbf{I} \varepsilon - (\beta + \beta') \mathbf{I} \varepsilon + L_C \end{aligned} \quad (20)$$

where \mathbf{I} denotes a column vector of ones and L_C is a function of $\boldsymbol{\psi}$.

Considering $\boldsymbol{\psi}^H \mathbf{R} \boldsymbol{\psi} = \boldsymbol{\psi}^H \text{Re}(\mathbf{R}) \boldsymbol{\psi}$, Eq.(20) can be rewritten in a more compact form as follows:

$$\begin{aligned} L_d = & -\frac{1}{2} \boldsymbol{\psi}^H \text{Re}(\mathbf{R} + \frac{\gamma}{2} \mathbf{I}) \boldsymbol{\psi} + \\ & \text{Re}[\boldsymbol{\psi}^H \mathbf{y}] - (\alpha + \alpha' + \beta + \beta') \mathbf{I} \varepsilon \end{aligned} \quad (21)$$

4 Performance analysis and simulation

In order to test the performance of the beamformer based on SVR scheme, a uniform linear array and a uniform circular

array have been considered respectively. The performance will be compared between conventional beamforming and SVR beamforming.

4.1 The beamforming of uniform linear arrays using SVR

Here, we first compared the algorithm with the conventional approach for a uniform linear array of 16 elements. The element spacing is $\lambda/2$. On the one hand, the desired signals with single source come from the angle of 50° with SNR -5dB . On the other hand, the desired signals with two non-coherent sources come from the angles of 50° and 100° with identical SNR -5dB .

For the above array, we used the Gaussian kernel function, fix the upper bound of Lagrange multipliers C in Eq.(6) to 10 and the factor of ε - insensitive loss function of Eq.(6) to 0.05, besides, we adopt a data set from only 64 points. Then we trained the beamformer which employs the algorithm with support vector regression approach.

We calculated the spatial spectrum of conventional approach and SVR for the same array. The simulation results can be seen in Fig.2 and Fig.3. In the following figures the dashed line denotes Conventional BeamForming without noise (abbreviated as CBF-S), the continuous line denotes Support Vector Regression BeamForming (abbreviated as BF-SVR) and the point dash line means Conventional BeamForming with noise (abbreviated as CBF-S+N).

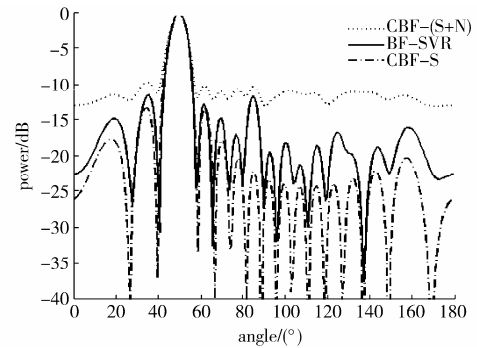


Fig.2 The beam pattern with single desired source

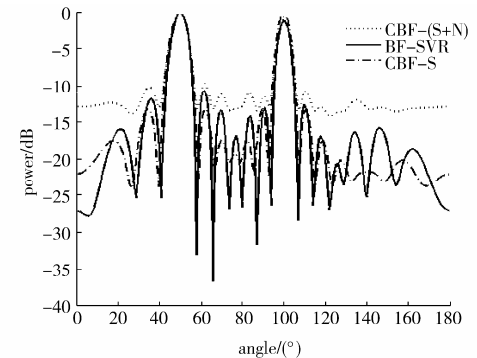


Fig. 3 The beam pattern with two non-coherent desired sources

4.2 The beamforming of uniform circular arrays using SVR

For a uniform circular array of 16 elements, we also applied the algorithm with conventional approach to the array, the radius of which is λ . On the one hand, the desired signals come from the angle of 50° with SNR -5dB . On the other hand, the desired signals with two non-coherent sources come from the angles of 50° and 100° with identical SNR -5dB .

Then according to the parameters of kernel function, the upper bound of Lagrange multipliers C and the factor of ε -insensitive loss function of Eq.(6) as stated in section 4.1, after adopting a data set from only 64 points, we trained the beamformer which employs the algorithm with support vector regression approach.

Finally, we calculated the spatial spectrum of conventional approach and SVR for the same array. The simulation results are shown in Fig.4 and Fig.5. As mentioned above, in the following figures the dashed line denotes Conventional BeamForming without noise (abbreviated as CBF-S), the continuous line denotes Support Vector Regression BeamForming (abbreviated as BF-SVR) and the point dash line means Conventional BeamForming with noise (abbreviated as CBF-S+N).

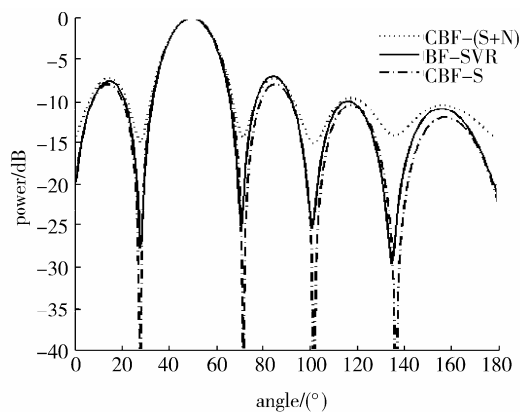


Fig. 4 The beam pattern with single desired source

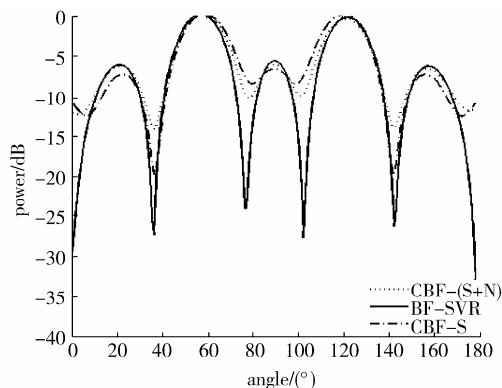


Fig.5 The beam pattern with two non-coherent desired sources

It can be seen from all the figures that without noise the curve BF-SVR plotted by a 64 points sampling way obtained with the SVR algorithm approximate the curve CBF-S by conventional beamforming method well. In other words, the result of SVR algorithm is close to the conventional beamforming method. Fig.2 and Fig.3 show the sidelobe level of linear arrays by the SVR algorithm can approximately reach -12dB , while those of circular arrays can approximately reach -9dB in the case of noise. Therefore, the sidelobe level of linear array and circular array using SVR approach in the circumstance of the desired signals with single source and two non-coherent sources is higher than regular beamforming.

5 Conclusions

Based on the overview of the mathematical foundations of the SVR algorithm, in this work an approach for optimizing beamforming, which is based on the Support Vector Regression (SVR), is introduced. After studying the mathematical principle of the SVR algorithm and its primal cost function, we firstly applied the modified cost function to the uniform array beamforming, and then optimized the corresponding parameters of the beamforming. The framework of SVR uniform array beamforming was then established. The examples illustrate the beamforming using SVR approach can approximate conventional beamforming well only when there's no noise and data set is small, which not only reduces the computation burden but also improves the generalization ability. In the low SNR the sidelobe level of linear arrays and circular arrays by the SVR algorithm is improved sharply than the conventional one. No matter either linear array or circular array, single source or two non-coherent sources are used, the beamforming which applies support vector regression is superior to the conventional beamforming. How to utilize the approach of nonlinear support vector regression for beamforming and to improve its performance need further research.

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