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Fully Nonlinear Shallow Water Waves Simulation Using Green-Naghdi Theory

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Abstract: Green-Naghdi (G-N) theory is a fully nonlinear theory for water waves. Some researchers call it a fully nonlinear Boussinesq model. Different degrees of complexity of G-N theory are distinguished by "levels" where the higher the level, the more complicated and presumably more accurate the theory is. In the research presented here a comparison was made between two different levels of G-N theory, specifically level II and level III G-N restricted theories. A linear analytical solution for level III G-N restricted theory was given. Waves on a planar beach and shoaling waves were both simulated with these two G-N theories. It was shown for the first time that level III G-N restricted theory can also be used to predict fluid velocity in shallow water. A level III G-N restricted theory is recommended instead of a level II G-N restricted theory when simulating fully nonlinear shallow water waves.

Keywords: Green-Naghdi theory; Boussinesq model; fully nonlinear water waves; shoaling waves
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1 Introduction

The simulation on fully nonlinear shallow water waves is extremely important for the design of coastal structures. The stream function wave theory (Rienecker and Fenton, 1981) can be used to simulate fully nonlinear water waves on a planar beach, but it's not so convenient for it to simulate shoaling waves. G-N models (Green *et al*., 1974) were first developed to analyze nonlinear free surface flows. The G-N approach is fundamentally different from the perturbation method based on developments in classical wave theory (Stokes wave theory and Boussinesq model). In the perturbation methods the field equation is satisfied exactly (the Laplacian of the velocity equals to zero) and the boundary conditions are approximated. In G-N theory, the opposite approach is used. That is, the kinematics of the field is assumed and the boundary conditions are satisfied exactly. For the kinematics, the velocity variation in the vertical and horizontal directions is taken to be simple polynomials of the vertical coordinate, where the coefficients of these polynomials can vary in time.

The G-N theory consists of an exact statement of the conservation of mass, an approximate statement of the conservation of momentum, and exact statements for various boundary conditions. These conditions yield coupled partial differential equations that can be reduced to several complicated governing equations by elimination of many of the variables. The final governing equations can be solved by Thomas algorithm (Demirbilek and Webster, 1992a). Experience with G-N theories (Demirbilek and Webster, 1992b) has shown that they can predict the shape and behavior of a wave up to almost the breaking limit. The G-N theory is very suitable for predicting the behavior of fully nonlinear water waves.

For deep water waves, the general method which is based on perturbation method was researched by Wang (2008). The fully nonlinear wave theory, G-N theory, was also applied to simulate irregular deep water waves (Webster and Kim, 1990). Zhao and Duan (2009) supplied fully nonlinear wave-maker boundary condition which improves the accuracy.

For shallow water waves, many Boussinesq models were developed. The fully nonlinear Boussinesq model with high dispersion accuracy was researched (Madsen *et al*., 2002; Madsen *et al*., 2003). But there are four-order derivatives and five-order derivatives in Boussinesq models, whose accuracy is hard to be guarantied. G-N model is another fully nonlinear Boussinesq model. The maximum derivative is only three-order. Level II G-N restricted theory where a linear variation of the horizontal velocity field in the vertical direction and a quadratic variation of the vertical velocity field is assumed was presented by Demirbilek and Webster (1992a). Different degrees of complexity of G-N theory are distinguished by "levels" where the higher the level, the more complicated and presumably more accurate the theory is. So, it is necessary to develop the G-N theory to level III.

In this paper, the Level III G-N restricted theory was investigated, and the linear analytical solution to Level III

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G-N restricted theory was given. The fully nonlinear wave making boundary condition based on stream function wave theory was used. The fully nonlinear waves on a planar beach were simulated by Level II and Level III G-N restricted theories. The velocity of fluid particles was calculated and compared by different level G-N theory for the first time. The shoaling waves were also simulated by using these two different G-N restricted theories. It is shown that Level III G-N restricted theory should be used to simulate fully nonlinear shallow water waves instead of Level II G-N restricted theory.

2 Governing equation for the G-N theory

We introduce a 2-D inertial Cartesian coordinate system with the *Oz* axis pointed vertically upwards and the *Ox* axis pointed horizontally to the right. The fluid horizontal and vertical velocity vector at a point (x,z) is denoted by $u(x,z,t), w(x,z,t)$. The free surface and the bottom are defined by $z=\beta(x,t)$ and $z=\alpha(x)$, respectively. This paper will be concerned only with an incompressible and inviscid fluid, and the mass density of the fluid ρ is constant.

The G-N theory introduces the following assumption (Demirbilek and Webster, 1992a):

$$\begin{cases} u(x,z,t)=\sum_{n=0}^{K}u_n(x,t)z^n \\ w(x,z,t)=\sum_{n=0}^{K}w_n(x,t)z^n \end{cases} \tag{1}$$

Using the assumption, we can obtain the governing equations for G-N restricted theory (Demirbilek and Webster, 1992a). The bottom kinematic boundary condition is

$$\sum_{n=0}^{K}w_n\alpha^n=\sum_{n=0}^{K}u_n\alpha^n\frac{\partial\alpha}{\partial x} \tag{2}$$

The free surface kinematic boundary condition can be expressed as

$$\sum_{n=0}^{K}w_n\beta^n=\frac{\partial\beta}{\partial t}+\sum_{n=0}^{K}u_n\beta^n\frac{\partial\beta}{\partial x} \tag{3}$$

The conservation of mass may be written as

$$u_K=0 \tag{4}$$

$$\frac{\partial u_n}{\partial x}+(n+1)w_{n+1}=0 \quad \text{for} \quad n=0,1,\cdots,K-1 \tag{5}$$

The conservation of momentum is found to be

$$\begin{cases} \sum_{m=0}^{K}(\rho\frac{\partial u_m}{\partial t}H_{m+n}+\sum_{r=0}^{K}\rho\frac{\partial u_m}{\partial x}u_rH_{m+r+n}+ \\ \sum_{r=0}^{K}\rho u_mw_rmH_{m+r+n-1})=-\frac{\partial P_n}{\partial x}+\hat{p}\beta^n\frac{\partial\beta}{\partial x}-\overline{p}\alpha^n\frac{\partial\alpha}{\partial x} \\ \sum_{m=0}^{K}(\rho\frac{\partial w_m}{\partial t}H_{m+n}+\sum_{r=0}^{K}\rho\frac{\partial w_m}{\partial x}u_rH_{m+r+n}+ \\ \sum_{r=0}^{K}\rho w_mw_rmH_{m+r+n-1})=P_n^{'}-\rho gH_n-\hat{p}\beta^n+\overline{p}\alpha^n \end{cases} \tag{6}$$

where $n=0,1,\cdots,K$, $H_n=\frac{1}{n+1}(\beta^{n+1}-\alpha^{n+1})$, $P_n=\int_\alpha^\beta pz^n\mathrm{d}z$, $P_n'=\int_\alpha^\beta pnz^{n-1}\mathrm{d}z$. $\overline{p}$ is the unknown pressure on the bottom, and $\hat{p}$ is the pressure on the free surface. If we exclude the surface tension effects, the value of $\hat{p}$ should be zero. As K increases, the complexity of G-N theory increases too. The case $K=1,2,3,\cdots$ indicates Level I, Level II, Level III$\cdots$ G-N restricted theory, respectively.

To further simplify the G-N equations, the reduction is done by expressing vertical components of the velocity in terms of horizontal components of the velocity. The expression is very long, so it is not shown here. Finally, we will get several coupled partial differential equations which can be solved by using a finite difference method named Thomas algorithm.

3 Linear analytical solution

3.1 Level II G-N restricted theory

The linear analytical solution to Level II G-N restricted theory had been given by Demirbilek and Webster (1992a), but they didn't present the derivation which will be given here. For level II G-N restricted theory, $K=2$. Using Eq.(1) and Eq.(4), the velocity variation in the vertical direction can be expressed as

$$\begin{cases} u=u_0+u_1z \\ w=w_0+w_1z+w_2z^2 \end{cases} \tag{7}$$

The elevation β and the coefficients u_0 and u_1 are the three unknowns in the final three equations for Level II G-N restricted theory. The water depth is assumed to be a constant here. The elevation β and the coefficients u_0 and u_1 are kept only first-order. Throw out the second-order terms and even higher terms in the three final equations, we find

$$\begin{cases} 60d\frac{\partial u_1}{\partial t}-120\frac{\partial u_0}{\partial t}-120g\frac{\partial\beta}{\partial t}+40d^2\frac{\partial^3u_0}{\partial x^2\partial t}-25d^3\frac{\partial^3u_1}{\partial x^2\partial t}=0 \\ 40d\frac{\partial u_1}{\partial t}-60\frac{\partial u_0}{\partial t}-60g\frac{\partial\beta}{\partial t}+25d^2\frac{\partial^3u_0}{\partial x^2\partial t}-16d^3\frac{\partial^3u_1}{\partial x^2\partial t}=0 \\ -\frac{\partial\beta}{\partial t}-d\frac{\partial u_0}{\partial x}+\frac{1}{2}d^2\frac{\partial u_1}{\partial x}=0 \end{cases} \tag{8}$$

Assuming the periodical solution of Eq.(8), the linear analytical solution to Level II G-N restricted theories can be expressed as

$$\begin{cases} \beta(x,t) = \beta_0 \cos[k(x-ct)] \\ u_0(x,t) = \dfrac{12g[20+7(kd)^2]\beta(x,t)}{c[240+104(kd)^2+3(kd)^4]} \\ u_1(x,t) = \dfrac{120gdk^2\beta(x,t)}{c[240+104(kd)^2+3(kd)^4]} \\ c^2 = \dfrac{24gd[(kd)^2+10]}{240+104(kd)^2+3(kd)^4} \end{cases} \tag{9}$$

where β_0 and d denote the amplitude and the water depth, respectively; k indicates the wave number and c denotes the wave celerity. By using g and k, we can obtain

$$\bar{c}^2 = \frac{24\bar{d}(\bar{d}^2+10)}{240+104\bar{d}^2+3\bar{d}^4} \tag{10}$$

Eq.(10) represents the non-dimensional linear dispersion relation for level II G-N restricted theory. The variable with a line on its head indicates the corresponding non-dimensional variables.

3.2 Level III G-N restricted theory

For level III G-N restricted theory, $K=3$. The velocity variation in the vertical direction can be expressed as

$$\begin{cases} u = u_0 + u_1 z + u_2 z^2 \\ w = w_0 + w_1 z + w_2 z^2 + w_3 z^3 \end{cases} \tag{11}$$

Using the similar method, we can obtain the linear analytical solution to Level III G-N restricted theory.

$$\begin{cases} \beta(x,t) = \beta_0 \cos[k(x-ct)] \\ u_0(x,t) = \dfrac{15g[420+192(kd)^2+7(kd)^4]\beta(x,t)}{c[6300+2880(kd)^2+135(kd)^4+(kd)^6]} \\ u_1(x,t) = \dfrac{30g(210dk^2+13d^3k^4)\beta(x,t)}{c[6300+2880(kd)^2+135(kd)^4+(kd)^6]} \\ u_2(x,t) = \dfrac{315gk^2[10+(kd)^2]\beta(x,t)}{c[6300+2880(kd)^2+135(kd)^4+(kd)^6]} \\ c^2 = \dfrac{15gd[420+52(kd)^2+(kd)^4]}{6300+2880(kd)^2+135(kd)^4+(kd)^6} \end{cases} \tag{12}$$

The non-dimensional linear dispersion relation for Level III G-N restricted theory is

$$\bar{c}^2 = \frac{15\bar{d}(420+52\bar{d}^2+\bar{d}^4)}{6300+2880\bar{d}^2+135\bar{d}^4+\bar{d}^6} \tag{13}$$

The non-dimensional linear dispersion relation for G-N theory can be compared with Stokes' result, the familiar dispersion relation from linear wave theory. In the same non-dimensional variables, this is given by

$$\bar{c}^2 = \tanh(\bar{d}) \tag{14}$$

Compared with the Stokes' linear result, the non-dimensional linear dispersion relation for Level III G-N restricted theory is better than Level II G-N restricted theory, as shown in Fig.1.

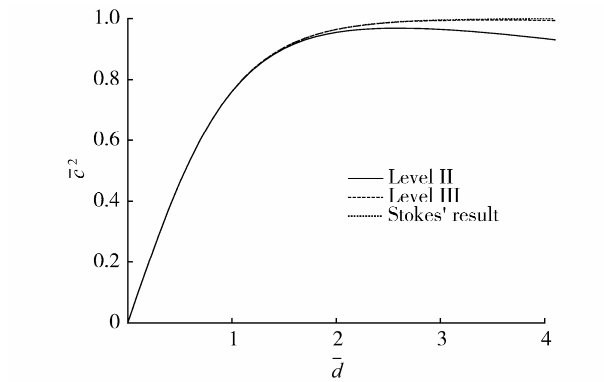


Fig.1 The non-dimensional linear dispersion relation

4 Boundary condition

The free surface boundary condition and the bottom boundary condition are met exactly in the G-N theory. The wave making boundary condition and the radiation boundary condition are discussed here in detail.

4.1 The wave making boundary condition

Different with former work by Demirbilek and Webster (1992a) who interpolate a series of tables for the wave making condition, a fully nonlinear wave making condition is proposed. For fully nonlinear shallow water waves on a planar beach, one can obtain the elevation and the velocity at any position by use of the stream function theory (Rienecker and Fenton, 1981). First, we set 101 vertical profiles equably in a wave length and set 101 points equably along every profile. For a vertical profile, the horizontal velocity of these 101 points can be determined by the stream function theory. According to the velocity assumption in the G-N theory, we can obtain the value of u_0, u_1, u_2 on this profile through least squares fitting. Similarly, we can obtain the value of u_0, u_1, u_2 on other profiles. Hence, we get the variation of u_0, u_1, u_2 in a wave period.

4.2 Radiation boundary condition

The Sommerfeld radiation boundary condition (Demirbilek and Webster, 1992a) is used here. It can be expressed as

$$\begin{cases} c\dfrac{\partial \beta}{\partial x} + \dfrac{\partial \beta}{\partial t} = 0 \\ c\dfrac{\partial u_n}{\partial x} + \dfrac{\partial u_n}{\partial t} = 0 \quad (n = 0,1,\cdots) \end{cases} \tag{15}$$

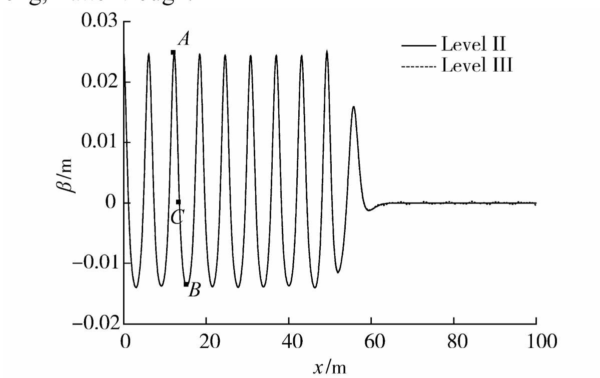
where c denotes the wave celerity. In practice, little reflection was observed, so the radiation boundary can be taken far enough away to minimize reflections.

5 Numerical computation and discussion

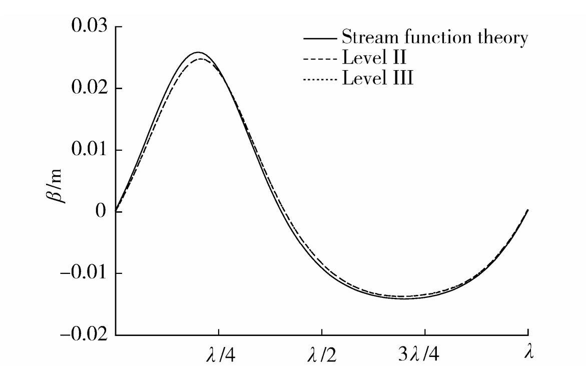
5.1 Water waves on a planar beach

A nonlinear water wave on a planar beach will be simulated by level II and Level III G-N restricted theory in this section. The water depth is 0.36 m, the wave period is 3.33 s, the wave height is 0.04 m, and the length of computational domain is 100 m.

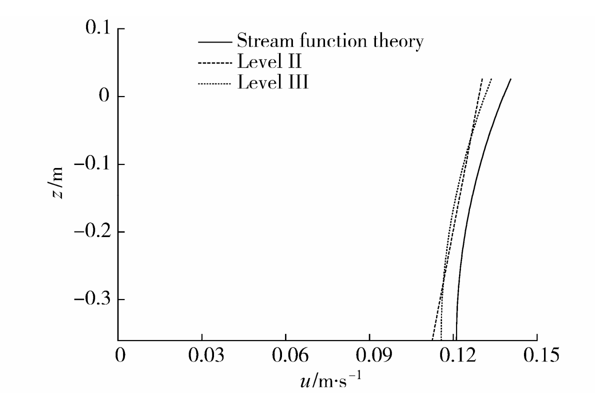
The results at 33.3 s are presented in Fig.2. Fig.2(a) shows the snapshot of the wave elevation at 33.3 s. The eldest waves (the rightmost two waves) are somewhat distorted, but regular waves follows. The stream function theory is used to compare with the results of G-N theory. The comparison of wave elevations is shown in Fig.2(b). The wave in Fig.2(b) has strong nonlinearity, so we find that it has sharp crest and long, flatter trough.



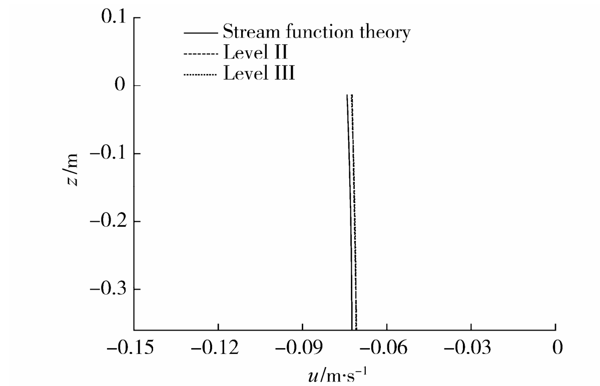
(a) The snapshot at 33.3 s



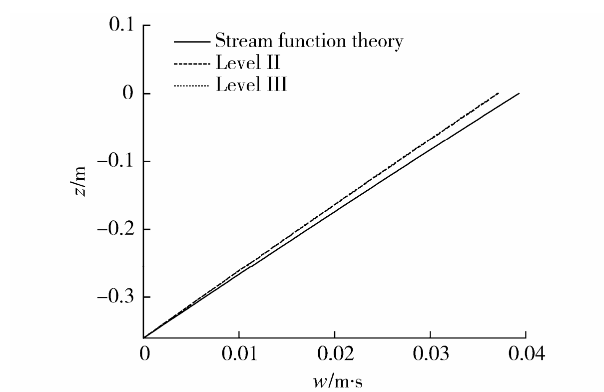
(b) Comparison with stream function theory



(c) The horizontal velocity at profile *A*



(d) The horizontal velocity at profile *B*



(e) The vertical velocity at profile *C*

Fig.2 Shallow water waves on a planar beach

There are three points, *A*, *B* and *C*, labeled in Fig.2(a). Four vertical profiles are chosen below the four points. The comparison of horizontal velocity is presented in Fig.2(c) and Fig.2(d). We can find that the results of G-N theory are close to the results of stream function wave theory, especially the Level III G-N restricted theory. It should be noted that the maximum horizontal velocity reaches 0.14 m/s in Fig.2(c), while the max horizontal velocity is only 0.8 m/s in Fig.2(d) because of the strong nonlinearity. The results of Level II G-N restricted theory show that the horizontal velocity variation along the water depth is just a linear function, it's not the fact. The result of Level III G-N restricted theory is better than Level II compared with the results of stream function theory.

The difference between G-N theory and stream function theory on horizontal velocity is shown in Fig.3. The maximum difference between Level III G-N theory and stream function theory is 4.9%, while the maximum difference between Level II G-N theory and stream function theory reaches 7.3%. The most important thing is that the results of Level III G-N theory have the same trend with stream function theory, and can be modified to agree with the stream function theory easily.

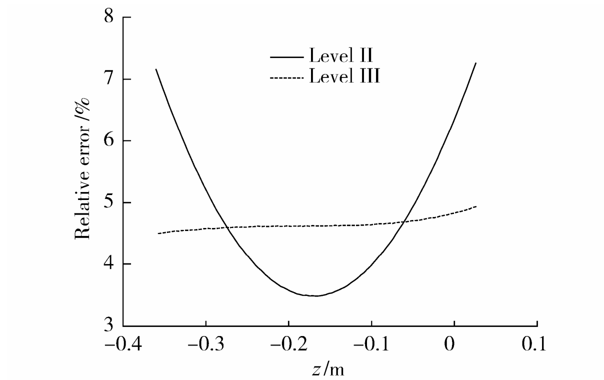


Fig.3 The relative error of horizontal velocity compared with stream function theory at profile *A*

The results of Level II and Level III G-N restricted theory are almost the same in the vertical velocity, and they are all close to the results of stream function theory.

5.2 Shoaling waves

The G-N theory is very suitable to deal with the situation that the seabed is arbitrary and irregular. In order to verify the results of G-N theory, the wave measured by Luth *et al*. (1994) is reproduced. The wave height is 0.02 m, the wave period is 2.02 s. The structure of the seabed is shown in Fig.4.

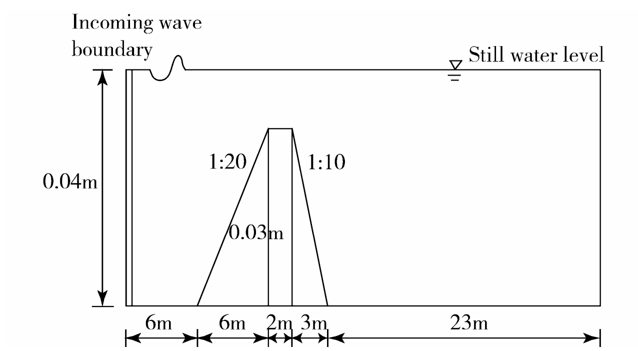
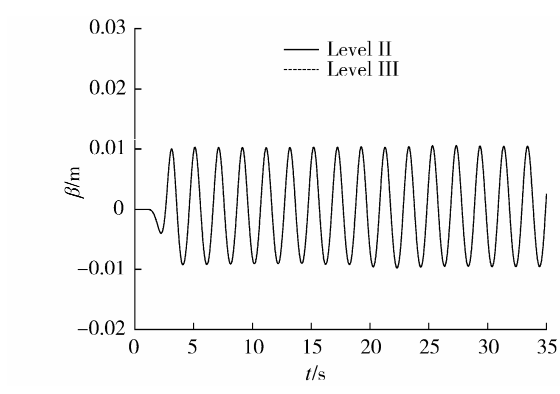
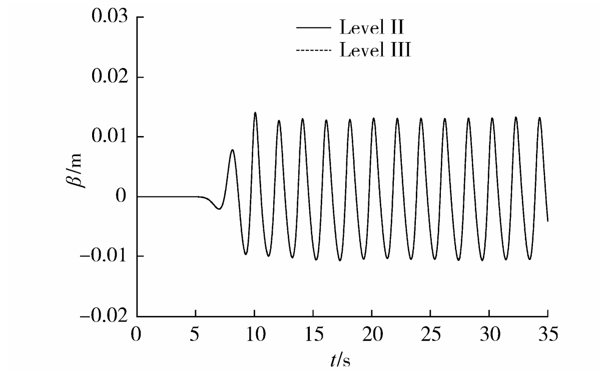


Fig.4 The structure of seabed

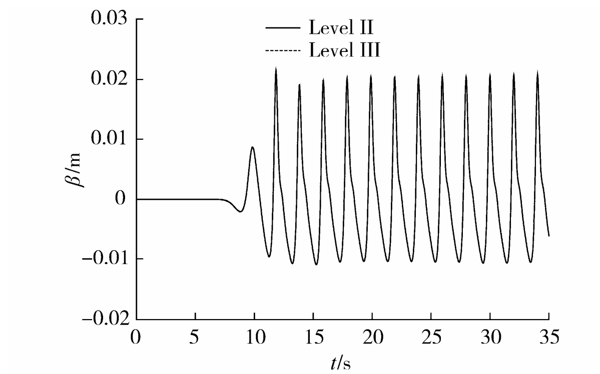
The time histories at six different positions (2.0, 10.5, 12.5, 13.5, 14.5, 19.0 m) are shown in Fig.5.



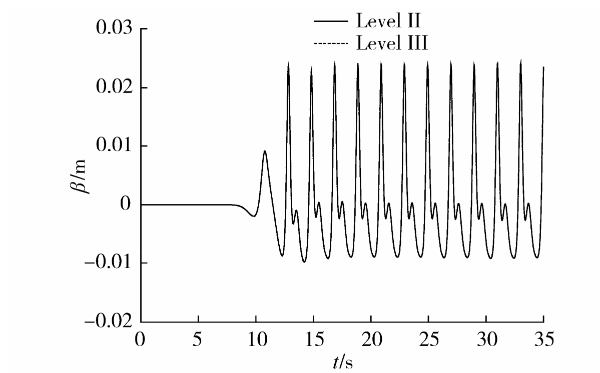
(a) x=2.0 m



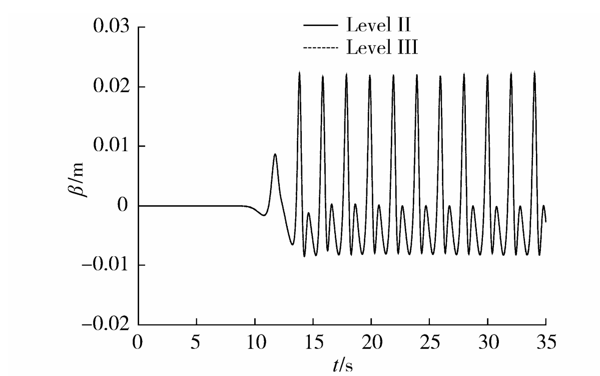
(b) x=10.5 m



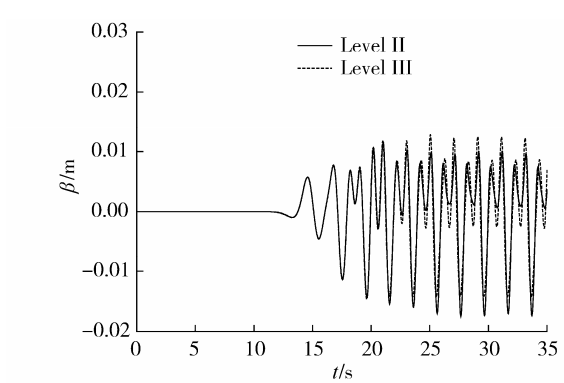
(c) x=12.5 m



(d) x=13.5 m



(e) x=14.5 m

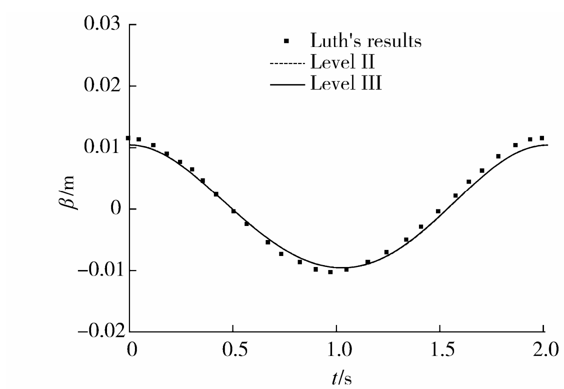


(f) x=19.0 m

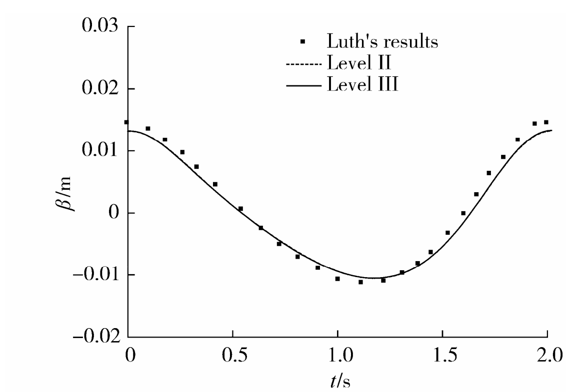
Fig.5 Time histories at six different positions

Initially, the wave crest is almost 0.01 m as shown in Fig.5(a). As the water depth becomes smaller, the water level rises. The wave crest can reaches 0.021 m at x=12.5 m, and even higher at x=13.5 m. This is the effect of shoaling water. When the wave tramps over the hill, the wave crest decreases because the water depth rises. When the wave spreads farther, the amplitude almost decreases to 0.01 m.

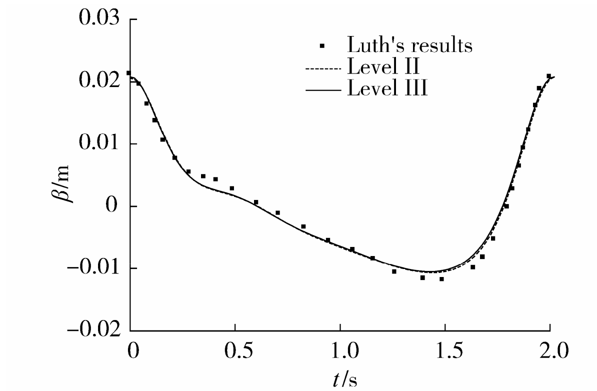
The results of wave elevation are compared with Luth's experimental results (Luth *et al*., 1994), as shown in Fig.6.



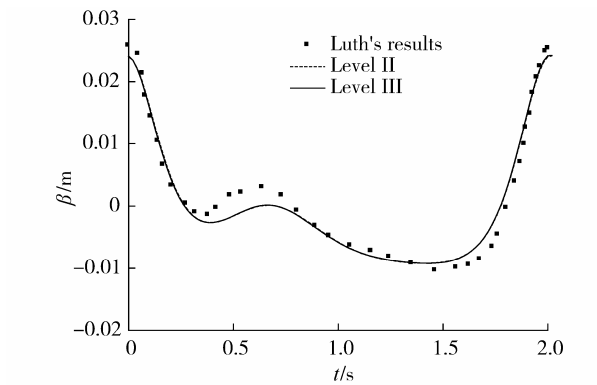
(a) x=2.0 m



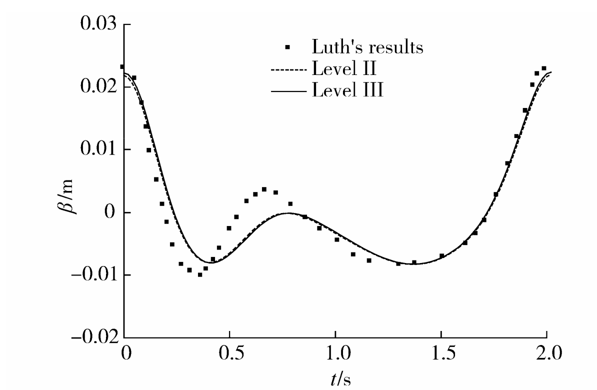
(b) x=10.5 m



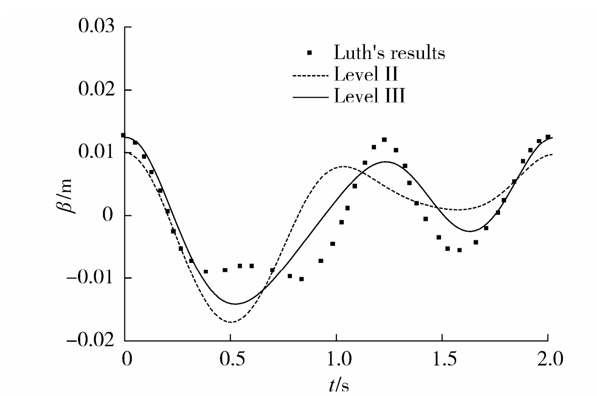
(c) x=12.5 m



(d) x=13.5 m



(e) x=14.5 m



(f) x=19.0 m

Fig.6 Comparison with stream function theory

In Fig.6, the results of Level II and Level III G-N theory are almost the same, and they are very close to the experimental results. The major difference can be seen at x=19.0 m from Fig.6(f). The result of Level III G-N restricted theory is better than Level II G-N restricted theory compared with Luth's experimental result.

6 Conclusions

Through simulating the fully nonlinear shallow water waves on a planar beach, it is found that fluid velocity from Level III G-N restricted theory matches well with the results of stream function theory. Hence, Level III G-N restricted theory can be used to predict both wave elevation and fluid velocity.

After comparing the linear analytical solution, the fluid velocity on a planar beach and the shoaling waves, it can be concluded that the results of Level III G-N restricted theory are better than Level II G-N restricted theory. So, Level III G-N restricted theory is recommended when simulating fully nonlinear shallow water waves.

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