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The asymptotic field of mode I quasi-static crack growth on the interface between a rigid and a pressure-sensitive material

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Abstract: A mechanical model of the quasi-static interface of a mode I crack between a rigid and a pressure-sensitive viscoelastic material was established to investigate the mechanical characteristic of ship-building engineering bi-materials. In the stable growth stage, stress and strain have the same singularity, ie $(\sigma,\varepsilon)\propto r^{-1/(n-1)}$. The variable-separable asymptotic solutions of stress and strain at the crack tip were obtained by adopting Airy's stress function and the numerical results of stress and strain in the crack-tip field were obtained by the shooting method. The results showed that the near-tip fields are mainly governed by the power-hardening exponent n and the Poisson ratio ν of the pressure-sensitive material. The fracture criterion of mode I quasi-static crack growth in pressure-sensitive materials, according to the asymptotic analyses of the crack-tip field, can be viewed from the perspective of strain.
Keywords: pressure-sensitive material; mode I quasi-static interface crack; crack-tip field; asymptotic analysis
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1 Introduction

With the development of material science, many engineering materials for ship-building are multi-material. The pressure-sensitive materials, such as polymers, ceramics and metallic glasses are characterized by pressure sensitive yielding and plastic dilatancy[1-3]. Hence, the researches on the elastic-plastic mechanical behavior of these materials have attracted much attention.

With regard to engineering materials, many researches have focused their attention on the interfaces because defects may occur easily in the interfacial regions and cause damage to the materials. Williams[4] was the first who analyzed a two-dimensional elastic problem of a crack lying along the interface between two dissimilar isotropic media by using eigenfunction method. Later, Erdogan[5], Rice and Sih[6-7] and Malyshev and Salganik[8] presented the similar problem by utilizing different methods. Shih and Asaro[9-11] originally investigated the solutions to an interface crack in elastic-plastic materials and gained a crack tip field similar to HRR field. An asymptotic solution to the rigid-viscoelastic interfacial crack-tip fields in bi-materials has been reported by Li Yong-dong, et al[12]. Tang Li-qiang[13] studied the crack-tip field of mode II stationary growth crack on elastic-elastic power law creeping bi-materials interface. The problem of a plane strain crack lying along an interface between a rigid substrate and elastic-plastic dilatant medium has been studied by Yuan[14]. Yu, et al[15] presented a plane-stress asymptotic field for a crack which lies along the interface of an elastic medium and a pressure-sensitive dilatant material.

The present work is aimed at studying the asymptotic fields for mode I quasi-static interfacial crack growth between the rigid material and pressure-sensitive material. To the end, by analyzing the index of the crack field, it is gotten that the stress and the strain have the same singularity in stable growing stage. The asymptotic solutions of the stress, the strain and the displacement that are of the variable-separable form are gained. The numerical results of the stress and the strain are obtained by shooting method and the influence which the material parameters have on the crack-tip field is discussed.

2 Governing equation

A Cartesian reference system is shown in Fig.1 with the origin o located at the crack tip. XOY is a fixed coordination and xoy is a moving coordination. For the sake of convenience, the polar system (r, θ) is also given. Material 1 is a pressure-sensitive material which is in the region of $0\le\theta\le\pi$, material 2 is a rigid material which

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is in the region of $-\pi \le \theta \le 0$. Assuming that V denotes the crack-tip's stable growth velocity, then $x = X - Vt$, $y = Y$, where t is the time.

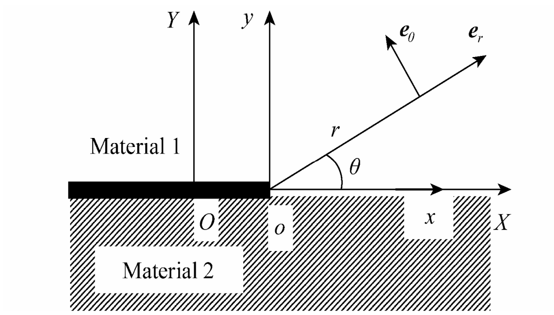


Fig.1 Mechanical model with interface crack

The material's derivative operator is

$$D = V\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} - \cos\theta\frac{\partial}{\partial r}\right]. \tag{1}$$

2.1 Basic equations

The constitutive equation for pressure-sensitive visco-elastic plastic materials under the plane strain condition is deduced:

$$\begin{aligned}\dot{\varepsilon}_{\alpha\beta} &= \frac{1}{E}\left[\dot{\sigma}_{\alpha\beta}(1+\nu) - \nu\dot{\sigma}_{kk}\delta_{\alpha\beta}\right] + \\ &B(t)\sigma_e^{n-1}\left[(1+\nu)\sigma_{\alpha\beta} - \nu\sigma_{kk}\delta_{\alpha\beta}\right],\end{aligned} \tag{2}$$

where E is the elastic modulus, n is the power harden index, $B(t)$ is the material parameter, and the effective stress is

$$\sigma_e^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 + (1-2\nu)\left(\frac{\sigma_x + \sigma_y}{2}\right)^2. \tag{3}$$

The relations of displacement-strain for small deformation are

$$\begin{cases}\varepsilon_r = \dfrac{\partial u_r}{\partial r}, \\ \varepsilon_\theta = \dfrac{u_r}{r} + \dfrac{\partial u_\theta}{r\partial\theta}, \\ \varepsilon_{r\theta} = \dfrac{1}{2}\left(\dfrac{\partial u_r}{r\partial\theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right).\end{cases} \tag{4}$$

The strain compatibility equation is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\dot{\varepsilon}_\theta) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(\dot{\varepsilon}_r) - \frac{1}{r}\frac{\partial}{\partial r}(\dot{\varepsilon}_r) - \frac{2}{r^2}\frac{\partial^2}{\partial r\partial\theta}(r\dot{\varepsilon}_{r\theta}) = 0. \tag{5}$$

For quasi-static growth crack, the inertia effect is negligible. Adopting Airy's stress function Ψ, let

$$\begin{cases}\sigma_r = \dfrac{1}{r}\dfrac{\partial\Psi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\Psi}{\partial\theta^2}, \\ \sigma_\theta = \dfrac{\partial^2\Psi}{\partial r^2}, \\ \sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial\Psi}{\partial\theta}\right).\end{cases} \tag{6}$$

Then the equilibrium equation comes into being automatically.

2.2 Asymptotic governing equation

Considering the singularity of the crack-tip field is of power law, when $r \to 0$, the stress function is expressed as

$$\Psi = \sigma_0 A r^{s_2} f(\theta), \tag{7}$$

where A is an undetermined parameter, $f(\theta)$ is the angular function without dimension, $\sigma_0 = \varepsilon_0 E$ is yield stress, and s_2 is an undetermined singularity.

The stress can be expressed as

$$\sigma_{\alpha\beta} = \sigma_0 A r^{s_2-2}\tilde{\sigma}_{\sigma\beta}, \tag{8}$$

where $\tilde{\sigma}_{\sigma\beta}$ is the angular distribution function of stress.

The effective stress is

$$\sigma_e = \sigma_0 A r^{s_2-2}\tilde{\sigma}_e. \tag{9}$$

Substituting Eq.(8) into Eq.(1), we get the stress-rate expression

$$\dot{\sigma}_{\alpha\beta} = V\sigma_0 A r^{s_2-3}\tilde{\dot{\sigma}}_{\sigma\beta}. \tag{10}$$

According to the singularity analyzed above, let

$$\boldsymbol{u} = \varepsilon_0 H r^{s_1-1}\tilde{\boldsymbol{u}}(\theta). \tag{11}$$

s_1 is an undetermined singularity index, $\varepsilon_0 = \sigma_0/E$, H is an undetermined parameter, $\tilde{\boldsymbol{u}}(\theta)$ is the angular function of displacement.

According to Eq.(1) and Eq.(4), the strain-rate in the variable-separable form can be expressed as

$$\dot{\varepsilon}_{\alpha\beta} = V\varepsilon_0 H r^{s_1-3}\tilde{\dot{\varepsilon}}_{\alpha\beta}. \tag{12}$$

In the stable growing stage, the elastic deformation and plastic deformation are equally dominant. According to the dimension compatibility, we get $s_1 = s_2 = (2n-3)/(n-1)$. Let $s = s_1 = s_2$. Then the stress and strain have the same singularity, ie $(\sigma,\varepsilon) \propto r^{-1/(n-1)}$. This is consistent with the singularity analysis of Taher[16].

According to Eq.(1), Eq.(4), Eq.(11) and Eq.(12), the components of the angular distribution function of strain-rate are expressed as

$$\begin{cases}\tilde{\dot{\varepsilon}}_r=(s-2)\tilde{u}_r'(\theta)\sin\theta-(s-2)(s-1)\tilde{u}_r(\theta)\cos\theta-(s-2)\tilde{u}_\theta(\theta)\sin\theta,\\ \tilde{\dot{\varepsilon}}_\theta=\tilde{u}_\theta''(\theta)\sin\theta-(s-2)\tilde{u}_\theta'(\theta)\cos\theta+(s-2)\tilde{u}_\theta(\theta)\sin\theta+2\tilde{u}_r'(\theta)\sin\theta-(s-2)\tilde{u}_r(\theta)\cos\theta,\\ \tilde{\dot{\varepsilon}}_{r\theta}=\dfrac{1}{2}[\tilde{u}_r''(\theta)\sin\theta-(s-2)\tilde{u}_r'(\theta)\cos\theta+(s-2)\tilde{u}_r(\theta)\sin\theta+(s-3)\tilde{u}_\theta'(\theta)\sin\theta-(s-2)^2\tilde{u}_\theta(\theta)\cos\theta].\end{cases}\tag{13}$$

According to Eq.(1), Eq.(6), Eq.(7), Eq.(8), Eq.(9) and Eq.(10), the components of the angular distribution function of stress are expressed as

$$\begin{cases}\tilde{\sigma}_r=f''+sf,\\ \tilde{\sigma}_\theta=s(s-1)f,\\ \tilde{\sigma}_{r\theta}=(1-s)f'.\end{cases}\tag{14}$$

The angular distribution function of effective stress is

$$\tilde{\sigma}_e=\left\{\frac{1}{4}\left[f''+(2-s)sf\right]^2+(1-s)^2f'^2+\frac{1}{4}(1-2\nu)\left(f''+s^2f\right)^2\right\}^{\frac{1}{2}}.\tag{15}$$

The components of the angular distribution function of stress-rate are expressed as

$$\begin{cases}\tilde{\dot{\sigma}}_r=f'''\sin\theta-f''(s-2)\cos\theta+\\ f'(3s-2)\sin\theta-f(s-2)s\cos\theta,\\ \tilde{\dot{\sigma}}_\theta=(s-2)(s-1)(f'\sin\theta-sf\cos\theta),\\ \tilde{\dot{\sigma}}_{r\theta}=f''(2-s)\sin\theta-(s-2)(1-s)f'\cos\theta+\\ \sin\theta(2-s)f.\end{cases}\tag{16}$$

Adopting a quantity without dimension $\dfrac{EB(t)}{V}\sigma_0^{n-1}A^{n-1}=1$, and letting $H=A$, substituting Eqs.(15)~(18) into Eq.(2), the governing equation can be expressed as

$$\begin{cases}f'''\sin\theta=f''(s-2)\cos\theta-f'(3s-2)\sin\theta+fs(s-2)\cos\theta+(s-2)\tilde{u}_r'(\theta)\sin\theta-\\ \quad(s-2)\tilde{u}_\theta\sin\theta-(s-2)(s-1)\tilde{u}_r(\theta)\cos\theta-\nu(s-2)(s-1)(f'\sin\theta-sf\cos\theta)+\\ \quad\tilde{\sigma}_e^{n-1}\left[f''+sf-\nu(s-1)sf\right],\\ \tilde{u}_\theta''(\theta)\sin\theta=(s-2)\tilde{u}_\theta'(\theta)\cos\theta-(s-2)(1-\nu)\tilde{u}_\theta(\theta)\sin\theta-[2+\nu(s-2)]\tilde{u}_r'(\theta)\sin\theta+\\ \quad(s-2)\left[1+\nu(s-1)\right]\tilde{u}_r(\theta)\cos\theta+(1-\nu^2)(s-2)(s-1)(f'\sin\theta-sf\cos\theta)+\\ \quad\tilde{\sigma}_e^{n-1}(1-\nu^2)(s-1)sf,\\ \tilde{u}_r''(\theta)\sin\theta=(s-2)\tilde{u}_r'(\theta)\cos\theta-(s-2)\tilde{u}_r(\theta)\sin\theta-(s-3)\tilde{u}_\theta'(\theta)\sin\theta+\\ \quad(s-2)^2\tilde{u}_\theta(\theta)\cos\theta+2(1+\nu)\left[f''(2-s)\sin\theta-(s-2)(1-s)f'\cos\theta+\sin\theta(2-s)f\right]+\\ \quad 2(1+\nu)\tilde{\sigma}_e^{n-1}(1-s)f'.\end{cases}\tag{17}$$

3 Solution conditions

The traction-free conditions on the crack surface require $\sigma_\theta|_{\theta=\pi}=\sigma_{r\theta}|_{\theta=\pi}=0$, which means

$$\begin{cases}f(\pi)=0,\\ f'(\pi)=0.\end{cases}\tag{18}$$

The continuity of tractions on the interface requires $\sigma_\theta|_\Gamma=\sigma_{r\theta}|_\Gamma=0$. For mode I crack, there is the condition of $\sigma_{r\theta}|_{\theta=0}=0$, then we obtain

$$f'(0)=0.\tag{19}$$

The continuity of displacement on the interface requires $u_\theta|_\Gamma=u_{r\theta}|_\Gamma=0$. As the material 2 is a rigid material, we have

$$\begin{cases}\tilde{u}_r(0)=0,\\ \tilde{u}_\theta(0)=0.\end{cases}\tag{20}$$

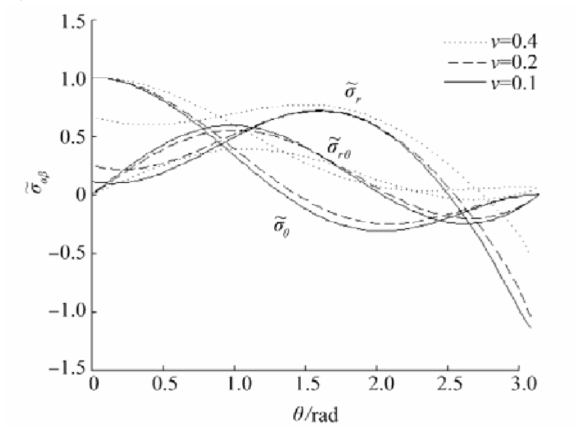
From Eq.(17), we obtain

$$
\begin{cases}
f''(0) = [\nu(s-1)-1]f(0)s, \\
\tilde{u}'_\theta(0) = (s-1)(1-\nu^2)sf(0) - \tilde{\sigma}_e^{n-1}(0)(1-\nu^2)s(s-1)f(0)\dfrac{1}{(s-2)}, \\
\tilde{u}'_r(0) = 0,
\end{cases}
\tag{21}
$$

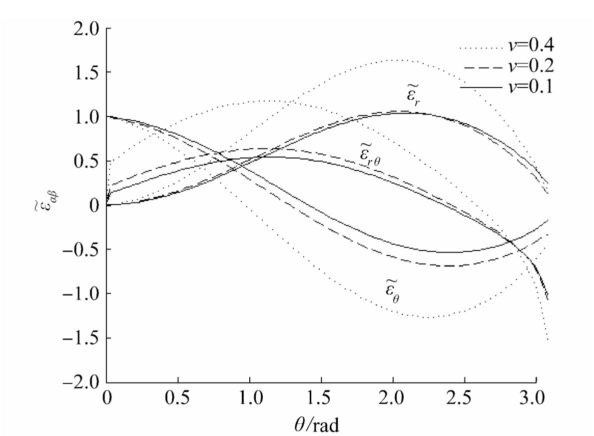
$$
\tilde{\sigma}_e(0) = \left\{ \frac{1}{4}\left[f''(0) + (2-s)sf(0) \right]^2 + \frac{1}{4}(1-2\nu)\left(f''(0) + s^2 f(0) \right)^2 \right\}^{\frac{1}{2}}.
$$

4 Numerical results and discussion

The above differential equations can be solved by shooting method. Then, the traction-free conditions are checked. In the calculation, the normalized conditions of $\max \tilde{\sigma}_\theta(\theta) = 1$ and $\max \tilde{\varepsilon}_\theta(\theta) = 1$ are adopted. The asymptotic fields of the stress and strain are shown in Figs.2~3. Under fixed value of n, the variation range of stress decreases as Poisson ratio increases and the variation range of strain increases as Poisson ratio increases. Under fixed value of Poisson ratio, the variation range of stress almost does not change as n increases, but the variation range of strain decreases as n increases.

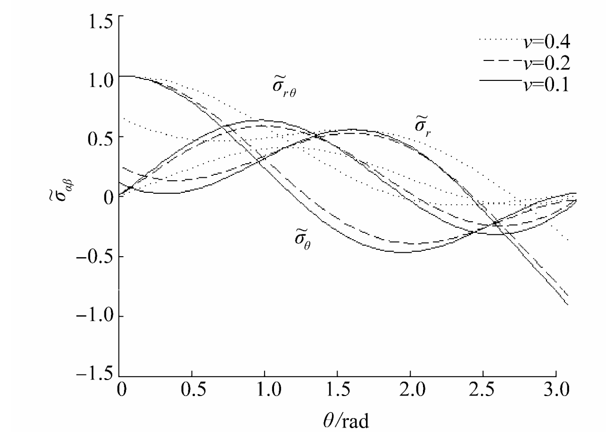


(a) Angular distributions of normalized stress

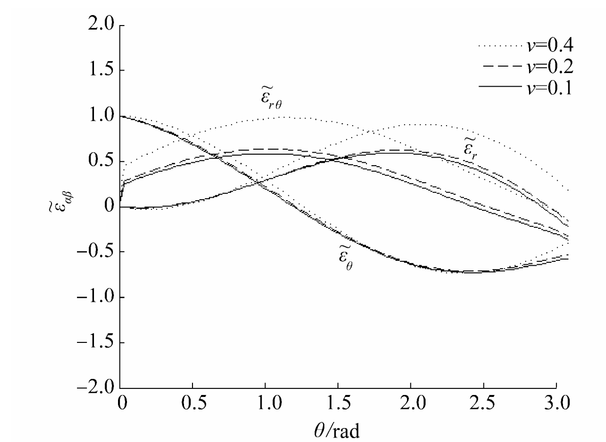


(b) Angular distributions of normalized strain

Fig.2 Angular distributions of stress and strain fields for different Poisson ratios in the case of n=5



(a) Angular distributions of normalized stress



(b) Angular distributions of normalized strain

Fig.3 Angular distributions of stress and strain fields for different Poisson ratios in the case of n=13

5 Conclusions

The important conclusions of this study are reached below.

1) The deduction and calculations of this study are only available in stable growing stage, while the stress and strain have the same singularity, namely $(\sigma, \varepsilon) \propto r^{-1/(n-1)}$, considering elastic deformation and plastic deformation are equally dominant in the near-tip field. Obviously,

when $n \to \infty$, the singularity of the stress and strain will vanish.

2) According to the asymptotic analysis of mode I quasi-static crack growth along the rigid and pressure-sensitive material interface, it is found that the hydrostatic stress at the crack tip is a tensile stress. Fig.2(a) and Fig.3(a) show that σ_θ, which gets to the maximum value at $\theta = 0$ and becomes the driving force of the crack stable growing.

3) The stress field near the crack tip is mainly governed by Possion ratio and less affected by the power-harden exponent *n*; the strain field near crack tip is mainly governed by Possion ratio and the power-harden exponent *n*, as the angular distributions of stress and strain fields shown in Figs.2~3. Clearly, the difference of stress or strain, in the case of $\nu = 0.1, 0.2$, is very small. But the difference of stress or strain between these and that in the case of $\nu = 0.4$ is much big. The variation ranges of stress and strain, in the case of $\nu = 0.4$, are also clearly much bigger than that in the case of other Possion ratios. This indicates that the material is more sensitive, and more easy to become failure in the case of $\nu = 0.4$. The variation ranges of strain decreases as the power-harden exponent *n* increases.

4) According to the asymptotic analysis of mode I quasi-static crack growth on the interface between a rigid and a pressure-sensitive material, as shown in Fig.2(b) and Fig.3(b), it is known that ε_θ gets to the maximum in the front of the crack. Then, the fracture criterion of mode I interface crack on the interface between a rigid and a pressere-sensitive material can be put forward from the perspective of strain, namely, $\varepsilon_\theta |_{r=r_0}^{\theta=\theta_0} = C$, where C is the material constant which can be determined by experiments or micromechanics.

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刚性－压力敏感性材料界面 I 型准静态扩展裂纹尖端场

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摘　要：为了研究船用工程复合材料的界面裂纹特性，建立了刚性－压力敏感粘弹塑性材料 I 型准静态扩展裂纹的力学模型. 在稳态扩展阶段，应力和应变具有相同的奇异量级，即 $(\sigma,\varepsilon)\propto r^{-1/(n-1)}$. 引入 Airy 应力函数，通过渐近分析得出了裂纹尖端应力和应变的分离变量形式的渐近解，并采用打靶法求得了裂纹尖端应力和应变的数值结果. 数值计算结果表明，界面裂尖场主要受材料的泊松比和幂硬化指数的控制. 通过对裂纹尖端场的渐近分析，从应变角度出发，提出了刚性－压力敏感性材料界面 I 型准静态扩展裂纹的断裂判据.

关键词：压力敏感性材料；界面 I 型准静态扩展裂纹；裂纹尖端场；渐近分析